Foundations of Neoclassical Growth

Ömer Özak

SMU

Macroeconomics II

Foundations of Neoclassical Growth

- Solow model: constant saving rate.
- More satisfactory to specify the *preference orderings* of individuals and derive their decisions from these preferences.
- Enables better understanding of the factors that affect savings decisions.
- Enables to discuss the "optimality" of equilibria
- Whether the (competitive) equilibria of growth models can be "improved upon".
- Notion of improvement: Pareto optimality.

Preliminaries I

- Consider an economy consisting of a unit measure of infinitely-lived households.
- I.e., an uncountable number of households: e.g., the set of households $\mathcal{H}$ could be represented by the unit interval $[0, 1]$.
- Emphasize that each household is infinitesimal and will have no effect on aggregates.
- Can alternatively think of $\mathcal{H}$ as a countable set of the form $\mathcal{H} = \{1, 2, ..., M\}$ with $M = \infty$, without any loss of generality.
- Advantage of unit measure: averages and aggregates are the same
- Simpler to have $\mathcal{H}$ as a finite set in the form $\{1, 2, ..., M\}$ with M large but finite.
- Acceptable for many models, but with overlapping generations require the set of households to be infinite.

Preliminaries II

- How to model households in infinite horizon?
  1. "infinitely lived" or consisting of overlapping generations with full altruism linking generations$\rightarrow$infinite planning horizon
  2. overlapping generations$\rightarrow$finite planning horizon (generally...).

Time Separable Preferences

- Standard assumptions on preference orderings so that they can be represented by utility functions.
- In particular, each household i has an *instantaneous utility function*

$$u_i\left(c_{it}\right),$$

- $u_i : \mathbb{R}_+ \to \mathbb{R}$ is increasing and concave and c_{it} is the consumption of household i in period t.
- Note instantaneous utility function is *not* specifying a complete preference ordering over all commodities—here consumption levels in all dates.
- Sometimes also referred to as the "felicity function".
- Two major assumptions in writing an instantaneous utility function
  1. consumption externalities are ruled out.
  2. overall utility is *time separable*.

Infinite Planning Horizon

- Start with the case of infinite planning horizon.
- Suppose households discount the future "exponentially"—or "proportionally".
- Interpret $u_i(\cdot)$ as a "Bernoulli utility function".
- Then preferences of household i at time $t = 0$ can be represented by a von Neumann-Morgenstern expected utility function.
- Thus household preferences at time $t = 0$ are

$$\mathbb{E}_0^i \sum_{t=0}^{\infty} \beta_i^t u_i(c_{it}), \tag{1}$$

where $\beta_i \in (0, 1)$ is the discount factor of household i.

Heterogeneity and the Representative Household

- $\mathbb{E}_0^i$ is the expectation operator with respect to the information set available to household i at time $t = 0$.
- So far index individual utility function, $u_i(\cdot)$, and the discount factor, β_i, by "i"
- Households could also differ according to their income processes. E.g., effective labor endowments of $\{e_{it}\}_{t=0}^{\infty}$, labor income of $\{e_{it} w_t\}_{t=0}^{\infty}$.
- But at this level of generality, this problem is not tractable.
- Follow the standard approach in macroeconomics and assume the existence of a *representative household*.

Time Consistency

- Exponential discounting and time separability: ensure "time-consistent" behavior.
- A solution $\{x_t\}_{t=0}^{T}$ (possibly with $T = \infty$) is *time consistent* if:
    - whenever $\{x_t\}_{t=0}^{T}$ is an optimal solution starting at time $t = 0$, $\{x_t\}_{t=t'}^{T}$ is an optimal solution to the continuation dynamic optimization problem starting from time $t = t' \in [0, T]$.

Challenges to the Representative Household

- An economy *admits a representative household* if preference side can be represented *as if* a single household made the aggregate consumption and saving decisions subject to a single budget constraint.
- This description concerning a representative household is purely positive
- Stronger notion of "normative" representative household: if we can also use the utility function of the representative household for welfare comparisons.
- Simplest case that will lead to the existence of a representative household: suppose each household is identical.

Representative Household II

- I.e., same β, same sequence $\{e_t\}_{t=0}^{\infty}$ and same

$$u\left(c_{it}\right)$$

  where $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is increasing and concave and c_{it} is the consumption of household i.

- Again ignoring uncertainty, preference side can be represented as the solution to

$$\max \sum_{t=0}^{\infty} \beta^t u\left(c_t\right), \tag{2}$$

- $\beta \in (0, 1)$ is the common discount factor and c_t the consumption level of the representative household.
- Admits a representative household rather trivially.
- Representative household's preferences, (2), can be used for positive and normative analysis.

Representative Household III

- If instead households are not identical but assume can model *as if* demand side generated by the optimization decision of a representative household...
- More realistic, but:
  1. The representative household will have positive, but not always a normative meaning.
  2. Models with heterogeneity: often do not lead to behavior that can be represented as if generated by a representative household.

Theorem **(Debreu-Mantel-Sonnenschein Theorem)** Let $\varepsilon > 0$ be a scalar and $N < \infty$ be a positive integer. Consider a set of prices $\boldsymbol{P}_{\varepsilon} = \left\{ p \in \mathbb{R}_{+}^{N} \colon \ p_j / p_{j'} \geq \varepsilon \text{ for all } j \text{ and } j' \right\}$ and any continuous function $\boldsymbol{x} : \boldsymbol{P}_{\varepsilon} \to \mathbb{R}_{+}^{N}$ that satisfies Walras' Law and is homogeneous of degree 0. Then there exists an exchange economy with N commodities and $H < \infty$ households, where the aggregate demand is given by $\boldsymbol{x}\left(p\right)$ over the set $\boldsymbol{P}_{\varepsilon}$.

Representative Household IV

- That excess demands come from optimizing behavior of households puts no restrictions on the form of these demands.
  - E.g., $\boldsymbol{x}(p)$ does not necessarily possess a negative-semi-definite Jacobian or satisfy the weak axiom of revealed preference (requirements of demands generated by individual households).
- Hence without imposing further structure, impossible to derive specific $\boldsymbol{x}(p)$'s from the maximization behavior of a single household.
- Severe warning against the use of the representative household assumption.
- Partly an outcome of very strong income effects:
  - special but approximately realistic preference functions, and restrictions on distribution of income rule out arbitrary aggregate excess demand functions.

Gorman Aggregation

- Recall an indirect utility function for household i, $v_i\left(p, y^i\right)$, specifies (ordinal) utility as a function of the price vector $p = (p_1, ..., p_N)$ and household's income y^i.
- $v_i\left(p, y^i\right)$: homogeneous of degree 0 in p and y.

Theorem **(Gorman's Aggregation Theorem)** Consider an economy with a finite number $N < \infty$ of commodities and a set $\mathcal{H}$ of households. Suppose that the preferences of household $i \in \mathcal{H}$ can be represented by an indirect utility function of the form

$$v^i\left(p, y^i\right) = a^i(p) + b(p)\, y^i, \tag{3}$$

then these preferences can be aggregated and represented by those of a representative household, with indirect utility

$$v(p, y) = \int_{i \in \mathcal{H}} a^i(p)\, di + b(p)\, y,$$

where $y \equiv \int_{i \in \mathcal{H}} y^i di$ is aggregate income.

Linear Engel Curves

- Demand for good j (from Roy's identity):

$$x_j^i\left(p, y^i\right) = -\frac{1}{b\left(p\right)}\frac{\partial a^i\left(p\right)}{\partial p_j} - \frac{1}{b\left(p\right)}\frac{\partial b\left(p\right)}{\partial p_j}y^i.$$

- Thus linear Engel curves.
- "Indispensable" for the existence of a representative household.
- Let us say that there exists a *strong representative household* if redistribution of income or endowments across households does not affect the demand side.
- Gorman preferences are sufficient for a strong representative household.
- Moreover, they are also *necessary* (with the same $b\left(p\right)$ for all households) for the economy to admit a strong representative household.
  - The proof is easy by a simple variation argument.

Importance of Gorman Preferences

- Gorman Preferences limit the **extent of income effects** and enables the aggregation of individual behavior.
- Integral is "Lebesgue integral," so when $\mathcal{H}$ is a finite or countable set, $\int_{i\in\mathcal{H}} y^i di$ is indeed equivalent to the summation $\sum_{i\in\mathcal{H}} y^i$.
- Stated for an economy with a finite number of commodities, but can be generalized for infinite or even a continuum of commodities.
- Note all we require is there exists a monotonic transformation of the indirect utility function that takes the form in (3)—as long as no uncertainty.
- Contains some commonly-used preferences in macroeconomics.

Example: Constant Elasticity of Substitution Preferences

- A very common class of preferences: constant elasticity of substitution (CES) preferences or Dixit-Stiglitz preferences.
- Suppose each household denoted by $i \in \mathcal{H}$ has total income y^i and preferences defined over $j = 1, ..., N$ goods

$$U^i\left(x_1^i, ..., x_N^i\right) = \left[\sum_{j=1}^{N}\left(x_j^i - \xi_j^i\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, \tag{4}$$

- $\sigma \in (0, \infty)$ and $\xi_j^i \in [-\bar{\xi}, \bar{\xi}]$ is a household specific term, which parameterizes whether the particular good is a necessity for the household.
- For example, $\xi_j^i > 0$ may mean that household i needs to consume a certain amount of good j to survive.

Example II

- If we define the level of consumption of each good as $\hat{x}_j^i = x_j^i - \xi_j^i$, the elasticity of substitution between any two $\hat{x}_j^i$ and $\hat{x}_{j'}^i$ would be equal to σ.
- Each consumer faces a vector of prices $p = (p_1, ..., p_N)$, and we assume that for all i,

$$\sum_{j=1}^{N} p_j \bar{\xi} < y^i,$$

- Thus household can afford a bundle such that $\hat{x}_j^i \geq 0$ for all j.
- The indirect utility function is given by

$$v^i\left(p, y^i\right) = \frac{\left[-\sum_{j=1}^{N} p_j \xi_j^i + y^i\right]}{\left[\sum_{j=1}^{N} p_j^{1-\sigma}\right]^{\frac{1}{1-\sigma}}}, \tag{5}$$

Example III

- Satisfies the Gorman form (and is also homogeneous of degree 0 in p and y).
- Therefore, this economy admits a representative household with indirect utility:

$$v(p, y) = \frac{\left[-\sum_{j=1}^{N} p_j \xi_j + y\right]}{\left[\sum_{j=1}^{N} p_j^{1-\sigma}\right]^{\frac{1}{1-\sigma}}}$$

- y is aggregate income given by $y \equiv \int_{i \in \mathcal{H}} y^i di$ and $\xi_j \equiv \int_{i \in \mathcal{H}} \xi_j^i di$.
- The utility function leading to this indirect utility function is

$$U(x_1, ..., x_N) = \left[\sum_{j=1}^{N} (x_j - \xi_j)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}. \tag{6}$$

- Preferences closely related to CES preferences will be key in ensuring *balanced growth* in neoclassical growth models.

Normative Representative Household

- Gorman preferences also imply the existence of a normative representative household.
- Recall an allocation is *Pareto optimal* if no household can be made strictly better-off without some other household being made worse-off.

Existence of Normative Representative Household

Theorem (Existence of a Normative Representative Household) Consider an economy with a finite number $N < \infty$ of commodities, a set $\mathcal{H}$ of households and a convex aggregate production possibilities set Y. Suppose that the preferences of each household $i \in \mathcal{H}$ take the Gorman form, $v^i\left(p, y^i\right) = a^i\left(p\right) + b\left(p\right) y^i$.

1. Then any allocation that maximizes the utility of the representative household, $v\left(p, y\right) = \sum_{i \in \mathcal{H}} a^i\left(p\right) + b\left(p\right) y$, with $y \equiv \sum_{i \in \mathcal{H}} y^i$, is Pareto optimal.
2. Moreover, if $a^i\left(p\right) = a^i$ for all p and all $i \in \mathcal{H}$, then any Pareto optimal allocation maximizes the utility of the representative household.

Proof of Theorem I

- Represent a Pareto optimal allocation as:

$$\max_{\{p_j\},\{y^i\},\{\mathbf{z}_j\}} \sum_{i\in\mathcal{H}} \alpha^i v^i\left(p, y^i\right) = \sum_{i\in\mathcal{H}} \alpha^i \left(a^i\left(p\right) + b\left(p\right) y^i\right)$$

subject to

$$\begin{aligned}
-\frac{1}{b\left(p\right)}\left(\sum_{i\in\mathcal{H}} \frac{\partial a^i\left(p\right)}{\partial p_j} + \frac{\partial b\left(p\right)}{\partial p_j} y\right) &= \mathbf{z}_j \in Y_j\left(p\right) \text{ for } j = 1, ..., N \\
\sum_{i\in\mathcal{H}} y^i &= y \equiv \sum_{j=1}^{N} p_j \mathbf{z}_j \\
\sum_{j=1}^{N} p_j \omega_j &= y, \\
p_j &\geq 0 \text{ for all } j.
\end{aligned}$$

Proof of Theorem II

- Here $\{\alpha^i\}_{i\in\mathcal{H}}$ are nonnegative Pareto weights with $\sum_{i\in\mathcal{H}}\alpha^i = 1$ and $\boldsymbol{z}_j \in Y_j(p)$ profit maximizing production of good j.
- First set of constraints use Roy's identity to express total demand for good j and set it equal to supply, z_j.
- Second equation sets value of income equal to value of production.
- Third equation makes sure total income is equal to the value of the endowments, ω_j.
- Compare the above maximization problem to:

$$\max \sum_{i\in\mathcal{H}} a^i(p) + b(p)\, y$$

  subject to the same set of constraints.
- The only difference is in the latter each household has been assigned the same weight.

Proof of Theorem III

- Let (p^*, y^*) be a solution to the second problem.
- By definition it is also a solution to the first problem with $\alpha^i = \alpha$, and therefore it is Pareto optimal.
- This establishes the first part of the theorem.
- To establish the second part, suppose that $a^i(p) = a^i$ for all p and all $i \in \mathcal{H}$.
- To obtain a contradiction, let $\boldsymbol{y} \in \mathbb{R}^{|\mathcal{H}|}$ and suppose that $(p_\alpha^{**}, \boldsymbol{y}_\alpha^{**})$ is a solution to the first problem for some weights $\{\alpha^i\}_{i \in \mathcal{H}}$ and suppose that it is not a solution to the second problem.

Proof of Theorem IV

- Let
$$\alpha^M = \max_{i \in \mathcal{H}} \alpha^i$$
and
$$\mathcal{H}^M = \{i \in \mathcal{H} \mid \alpha^i = \alpha^M\}$$
be the set of households given the maximum Pareto weight.
- Let (p^*, y^*) be a solution to the second problem such that
$$y^i = 0 \text{ for all } i \notin \mathcal{H}^M. \tag{7}$$
- Such a solution exists since objective function and constraint set in the second problem depend only on the vector $(y^1, .., y^{|\mathcal{H}|})$ through $y = \sum_{i \in \mathcal{H}} y^i$.

Proof of Theorem V

- Since, by definition, $\left(p_{\alpha}^{**}, \mathbf{y}_{\alpha}^{**}\right)$ is in the constraint set of the second problem and is not a solution,

$$\begin{aligned} \sum_{i\in\mathcal{H}} a^i + b\left(p^*\right)y^* &> \sum_{i\in\mathcal{H}} a^i + b\left(p_{\alpha}^{**}\right)y_{\alpha}^{**} \\ b\left(p^*\right)y^* &> b\left(p_{\alpha}^{**}\right)y_{\alpha}^{**}. \end{aligned} \tag{8}$$

- The hypothesis that it is a solution to the first problem also implies

$$\begin{aligned} \sum_{i\in\mathcal{H}} \alpha^i a^i + \sum_{i\in\mathcal{H}} \alpha^i b\left(p_{\alpha}^{**}\right)\left(y_{\alpha}^{**}\right)^i &\geq \sum_{i\in\mathcal{H}} \alpha^i a^i + \sum_{i\in\mathcal{H}} \alpha^i b\left(p^*\right)\left(y^*\right)^i \\ \sum_{i\in\mathcal{H}} \alpha^i b\left(p_{\alpha}^{**}\right)\left(y_{\alpha}^{**}\right)^i &\geq \sum_{i\in\mathcal{H}} \alpha^i b\left(p^*\right)\left(y^*\right)^i . \end{aligned} \tag{9}$$

- Then, it can be seen that any solution $\left(p^{**}, y^{**}\right)$ to the Pareto optimal allocation problem satisfies $y^i = 0$ for any $i \notin \mathcal{H}^M$.

Proof of Theorem VI

- In view of this and the choice of (p^*, y^*) in (7), equation (9) implies

$$\begin{aligned}
\alpha^M b(p_\alpha^{**}) \sum_{i\in\mathcal{H}} (y_\alpha^{**})^i &\geq \alpha^M b(p^*) \sum_{i\in\mathcal{H}} (y^*)^i \\
b(p_\alpha^{**})(y_\alpha^{**}) &\geq b(p^*)(y^*),
\end{aligned}$$

- Contradicts equation (8): hence under the stated assumptions, any Pareto optimal allocation maximizes the utility of the representative household.

Infinite Planning Horizon I

- Most growth and macro models assume that individuals have an infinite-planning horizon
- Two reasonable microfoundations for this assumption
- First: "Poisson death model" or the *perpetual youth model*: individuals are finitely-lived, but not aware of when they will die.
    1. Strong simplifying assumption: likelihood of survival to the next age in reality is not a constant
    2. But a good starting point, tractable and implies expected lifespan of $1/\nu < \infty$ periods, can be used to get a sense value of ν.
- Suppose each individual has a standard instantaneous utility function $u : \mathbb{R}_+ \to \mathbb{R}$, and a "true" or "pure" discount factor $\hat{\beta}$
- Normalize $u(0) = 0$ to be the utility of death.
- Consider an individual who plans to have a consumption sequence $\{c_t\}_{t=0}^{\infty}$ (conditional on living).

Infinite Planning Horizon II

- Individual would have an *expected* utility at time $t = 0$ given by

$$\begin{aligned} U(0) &= u(c_0) + \hat{\beta}(1-\nu)u(c_1) + \hat{\beta}\nu u(0) \\ &\quad + \hat{\beta}^2(1-\nu)^2 u(c_2) + \hat{\beta}^2(1-\nu)\nu u(0) + \ldots \\ &= \sum_{t=0}^{\infty} \left(\hat{\beta}(1-\nu)\right)^t u(c_t) \\ &= \sum_{t=0}^{\infty} \beta^t u(c_t), \end{aligned} \tag{10}$$

- Second line collects terms and uses $u(0) = 0$, third line defines $\beta \equiv \hat{\beta}(1-\nu)$ as "effective discount factor."
- Isomorphic to model of infinitely-lived individuals, but values of β may differ.
- Also equation (10) is already the expected utility; probabilities have been substituted.

Infinite Planning Horizon III

- Second: intergenerational altruism or from the "bequest" motive.
- Imagine an individual who lives for one period and has a single offspring (who will also live for a single period and beget a single offspring etc.).
- Individual not only derives utility from his consumption but also from the bequest he leaves to his offspring.
- For example, utility of an individual living at time t is given by

$$u\left(c_t\right) + U^b\left(b_t\right),$$

- c_t is his consumption and b_t denotes the bequest left to his offspring.
- For concreteness, suppose that the individual has total income y_t, so that his budget constraint is

$$c_t + b_t \leq y_t.$$

Infinite Planning Horizon IV

- $U^b(\cdot)$: how much the individual values bequests left to his offspring.
- Benchmark might be "purely altruistic:" cares about the utility of his offspring (with some discount factor).
- Let discount factor between generations be β.
- Assume offspring will have an income of w without the bequest.
- Then the utility of the individual can be written as

$$u(c_t) + \beta V(b_t + w),$$

- $V(\cdot)$: continuation value, the utility that the offspring will obtain from receiving a bequest of b_t (plus his own w).
- Value of the individual at time t can in turn be written as

$$V(y_t) = \max_{c_t + b_t \leq y_t} \{u(c_t) + \beta V(b_t + w_{t+1})\},$$

Infinite Planning Horizon V

- Canonical form of a dynamic programming representation of an infinite-horizon maximization problem.
- Under some mild technical assumptions, this dynamic programming representation is equivalent to maximizing

$$\sum_{s=0}^{\infty} \beta^s u\left(c_{t+s}\right)$$

  at time t.
- Each individual internalizes utility of all future members of the "dynasty".
- Fully altruistic behavior within a dynasty ("dynastic" preferences) will also lead to infinite planning horizon.

The Representative Firm I

- While not all economies would admit a representative household, standard assumptions (in particular no production externalities and competitive markets) are sufficient to ensure a representative firm.

Theorem **(The Representative Firm Theorem)** Consider a competitive production economy with $N \in \mathbb{N} \cup \{+\infty\}$ commodities and a countable set $\mathcal{F}$ of firms, each with a convex production possibilities set $Y^f \subset \mathbb{R}^N$. Let $p \in \mathbb{R}_+^N$ be the price vector in this economy and denote the set of profit maximizing net supplies of firm $f \in \mathcal{F}$ by $\hat{Y}^f(p) \subset Y^f$ (so that for any $\hat{y}^f \in \hat{Y}^f(p)$, we have $p \cdot \hat{y}^f \geq p \cdot y^f$ for all $y^f \in Y^f$). Then there exists a *representative firm* with production possibilities set $Y \subset \mathbb{R}^N$ and set of profit maximizing net supplies $\hat{Y}(p)$ such that for any $p \in \mathbb{R}_+^N$, $\hat{y} \in \hat{Y}(p)$ if and only if $\hat{y}(p) = \sum_{f \in \mathcal{F}} \hat{y}^f$ for some $\hat{y}^f \in \hat{Y}^f(p)$ for each $f \in \mathcal{F}$.

Proof of Theorem: The Representative Firm I

- Let Y be defined as follows:

$$Y = \left\{ \sum_{f \in \mathcal{F}} y^f : y^f \in Y^f \text{ for each } f \in \mathcal{F} \right\}.$$

- To prove the "if" part of the theorem, fix $p \in \mathbb{R}_+^N$ and construct $\hat{y} = \sum_{f \in \mathcal{F}} \hat{y}^f$ for some $\hat{y}^f \in \hat{Y}^f(p)$ for each $f \in \mathcal{F}$.
- Suppose, to obtain a contradiction, that $\hat{y} \notin \hat{Y}(p)$, so that there exists y' such that $p \cdot y' > p \cdot \hat{y}$.

Proof of Theorem: The Representative Firm II

- By definition of the set Y, this implies that there exists $\left\{y^f\right\}_{f\in\mathcal{F}}$ with $y^f \in Y^f$ such that

$$
\begin{aligned}
p\cdot\left(\sum_{f\in\mathcal{F}} y^f\right) &> p\cdot\left(\sum_{f\in\mathcal{F}} \hat{y}^f\right) \\
\sum_{f\in\mathcal{F}} p\cdot y^f &> \sum_{f\in\mathcal{F}} p\cdot \hat{y}^f,
\end{aligned}
$$

  so that there exists at least one $f' \in \mathcal{F}$ such that

$$p\cdot y^{f'} > p\cdot \hat{y}^{f'},$$

- Contradicts the hypothesis that $\hat{y}^f \in \hat{Y}^f(p)$ for each $f \in \mathcal{F}$ and completes this part of the proof.

Proof of Theorem: The Representative Firm III

- To prove the "only if" part of the theorem, let $\hat{y} \in \hat{Y}(p)$ be a profit maximizing choice for the representative firm.
- Then, since $\hat{Y}(p) \subset Y$, we have that

$$\hat{y} = \sum_{f \in \mathcal{F}} y^f$$

for some $y^f \in Y^f$ for each $f \in \mathcal{F}$.

- Let $\hat{y}^f \in \hat{Y}^f(p)$. Then,

$$\sum_{f \in \mathcal{F}} p \cdot y^f \leq \sum_{f \in \mathcal{F}} p \cdot \hat{y}^f,$$

which implies that

$$p \cdot \hat{y} \leq p \cdot \sum_{f \in \mathcal{F}} \hat{y}^f. \tag{11}$$

Proof of Theorem: The Representative Firm IV

- Since, by hypothesis, $\sum_{f\in\mathcal{F}} \hat{y}^f \in Y$ and $\hat{y} \in \hat{Y}(p)$, we also have

$$p \cdot \hat{y} \geq p \cdot \sum_{f\in\mathcal{F}} \hat{y}^f.$$

- Therefore, inequality (11) must hold with equality, so that

$$p \cdot y^f = p \cdot \hat{y}^f,$$

for each $f \in \mathcal{F}$, and thus $y^f \in \hat{Y}^f(p)$. This completes the proof of the theorem.

The Representative Firm II

- Why such a difference between representative household and representative firm assumptions? Income effects.
- Changes in prices create income effects, which affect different households differently.
- No income effects in producer theory, so the representative firm assumption is without loss of any generality.
- Does not mean that heterogeneity among firms is uninteresting or unimportant.
- Many models of endogenous technology feature productivity differences across firms, and firms' attempts to increase their productivity relative to others will often be an engine of economic growth.

Problem Formulation I

- Discrete time infinite-horizon economy and suppose that the economy admits a representative household.
- Once again ignoring uncertainty, the representative household has the $t = 0$ objective function

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \tag{12}$$

with a discount factor of $\beta \in (0, 1)$.

- In continuous time, this utility function of the representative household becomes

$$\int_0^{\infty} \exp(-\rho t)\, u(c(t))\, dt \tag{13}$$

where $\rho > 0$ is now the discount rate of the individuals.

Problem Formulation II

- Where does the exponential form of the discounting in (13) come from?
- Calculate the value of \$1 in T periods, and divide the interval $[0, T]$ into $T/\Delta t$ equally-sized subintervals.
- Let the interest rate in each subinterval be equal to $\Delta t \cdot r$.
- Key: r is multiplied by Δt, otherwise as we vary Δt, we would be changing the interest rate.
- Using the standard compound interest rate formula, the value of \$1 in T periods at this interest rate is

$$v\left(T \mid \Delta t\right) \equiv \left(1 + \Delta t \cdot r\right)^{T/\Delta t}.$$

- Now we want to take the continuous time limit by letting $\Delta t \to 0$,

$$v\left(T\right) \equiv \lim_{\Delta t \to 0} v\left(T \mid \Delta t\right) \equiv \lim_{\Delta t \to 0} \left(1 + \Delta t \cdot r\right)^{T/\Delta t}.$$

Problem Formulation III

- Thus

$$
\begin{aligned}
v(T) &\equiv \exp\left[\lim_{\Delta t \to 0} \ln(1+\Delta t \cdot r)^{T/\Delta t}\right] \\
&= \exp\left[\lim_{\Delta t \to 0} \frac{T}{\Delta t} \ln(1+\Delta t \cdot r)\right].
\end{aligned}
$$

- The term in square brackets has a limit on the form $0/0$.
- Write this as and use L'Hospital's rule:

$$
\lim_{\Delta t \to 0} \frac{\ln(1+\Delta t \cdot r)}{\Delta t / T} = \lim_{\Delta t \to 0} \frac{r/(1+\Delta t \cdot r)}{1/T} = rT,
$$

- Therefore,

$$
v(T) = \exp(rT).
$$

- Conversely, \$1 in T periods from now, is worth $\exp(-rT)$ today.
- Same reasoning applies to utility: utility from $c(t)$ in t evaluated at time 0 is $\exp(-\rho t)\, u(c(t))$, where ρ is (subjective) discount rate.

Welfare Theorems I

- There should be a close connection between Pareto optima and competitive equilibria.
- Start with models that have a finite number of consumers, so $\mathcal{H}$ is finite.
- However, allow an infinite number of commodities.
- Results here have analogs for economies with a continuum of commodities, but focus on countable number of commodities.
- Let commodities be indexed by $j \in \mathbb{N}$ and $x^i \equiv \left\{x_j^i\right\}_{j=0}^{\infty}$ be the consumption bundle of household i, and $\omega^i \equiv \left\{\omega_j^i\right\}_{j=0}^{\infty}$ be its endowment bundle.
- Assume feasible x^i's must belong to some consumption set $X^i \subset \mathbb{R}_+^{\infty}$.
- Most relevant interpretation for us is that at each date $j = 0, 1, \ldots$, each individual consumes a finite dimensional vector of products.

Welfare Theorems II

- Thus $x_j^i \in X_j^i \subset \mathbb{R}_+^K$ for some integer K.
- Consumption set introduced to allow cases where individual may not have negative consumption of certain commodities.
- Let $\boldsymbol{X} \equiv \prod_{i \in \mathcal{H}} X^i$ be the Cartesian product of these consumption sets, the aggregate consumption set of the economy.
- Also use the notation $\boldsymbol{x} \equiv \{x^i\}_{i \in \mathcal{H}}$ and $\boldsymbol{\omega} \equiv \{\omega^i\}_{i \in \mathcal{H}}$ to describe the entire consumption allocation and endowments in the economy.
- Feasibility requires that $\boldsymbol{x} \in \boldsymbol{X}$.
- Each household in $\mathcal{H}$ has a well defined preference ordering over consumption bundles.
- This preference ordering can be represented by a relationship $\succsim_i$ for household i, such that $x' \succsim_i x$ implies that household i weakly prefers $\boldsymbol{x}'$ to $\boldsymbol{x}$.

Welfare Theorems III

- Suppose that preferences can be represented by $u^i : X^i \to \mathbb{R}$, such that whenever $x' \succsim_i x$, we have $u^i(x') \geq u^i(x)$.
- The domain of this function is $X^i \subset \mathbb{R}_+^\infty$.
- Let $\boldsymbol{u} \equiv \left\{u^i\right\}_{i\in\mathcal{H}}$ be the set of utility functions.
- Production side: finite number of firms represented by $\mathcal{F}$
- Each firm $f \in \mathcal{F}$ is characterized by production set Y^f, specifies levels of output firm f can produce from specified levels of inputs.
- I.e., $y^f \equiv \left\{y_j^f\right\}_{j=0}^\infty$ is a feasible production plan for firm f if $y^f \in Y^f$.
- E.g., if there were only labor and a final good, Y^f would include pairs $(-l, y)$ such that with labor input l the firm can produce at most y.

Welfare Theorems IV

- Take each Y^f to be a *cone*, so that if $y \in Y^f$, then $\lambda y \in Y^f$ for any $\lambda \in \mathbb{R}_+$. This implies:
  1. $0 \in Y^f$ for each $f \in \mathcal{F}$;
  2. each Y^f exhibits constant returns to scale.
- If there are diminishing returns to scale from some scarce factors, this is added as an additional factor of production and Y^f is still a cone.
- Let $\boldsymbol{Y} \equiv \prod_{f \in \mathcal{F}} Y^f$ represent the aggregate production set and $\boldsymbol{y} \equiv \left\{y^f\right\}_{f \in \mathcal{F}}$ such that $y^f \in Y^f$ for all f, or equivalently, $\boldsymbol{y} \in \boldsymbol{Y}$.
- Ownership structure of firms: if firms make profits, they should be distributed to some agents
- Assume there exists a sequence of numbers (profit shares) $\boldsymbol{\theta} \equiv \left\{\theta_f^i\right\}_{f \in \mathcal{F}, i \in \mathcal{H}}$ such that $\theta_f^i \geq 0$ for all f and i, and $\sum_{i \in \mathcal{H}} \theta_f^i = 1$ for all $f \in \mathcal{F}$.
- θ_f^i is the share of profits of firm f that will accrue to household i.

Welfare Theorems V

- An economy $\mathcal{E}$ is described by $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \boldsymbol{u}, \boldsymbol{\omega}, \boldsymbol{Y}, \boldsymbol{X}, \boldsymbol{\theta})$.
- An allocation $(\boldsymbol{x}, \boldsymbol{y})$ is *feasible* if, and only if, $\boldsymbol{x} \in \boldsymbol{X}$, $\boldsymbol{y} \in \boldsymbol{Y}$, and $\sum_{i \in \mathcal{H}} x_j^i \leq \sum_{i \in \mathcal{H}} \omega_j^i + \sum_{f \in \mathcal{F}} y_j^f$ for all $j \in \mathbb{N}$.
- A price system is a sequence $p \equiv \{p_j\}_{j=0}^{\infty}$, such that $p_j \geq 0$ for all j.
- We can choose one of these prices as the numeraire and normalize it to 1.
- Also define $p \cdot x$ as the inner product of p and x, i.e., $p \cdot x \equiv \sum_{j=0}^{\infty} p_j x_j$.

Welfare Theorems VI

Definition A competitive equilibrium for the economy $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \boldsymbol{u}, \boldsymbol{\omega}, \boldsymbol{Y}, \boldsymbol{X}, \boldsymbol{\theta})$ is given by an allocation $\left(\boldsymbol{x}^* = \{x^{i*}\}_{i \in \mathcal{H}}, \boldsymbol{y}^* = \{y^{f*}\}_{f \in \mathcal{F}}\right)$ and a price system p^* such that

1. The allocation $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is feasible, i.e., $x^{i*} \in X^i$ for all $i \in \mathcal{H}$, $y^{f*} \in Y^f$ for all $f \in \mathcal{F}$ and
$$\sum_{i \in \mathcal{H}} x_j^{i*} \leq \sum_{i \in \mathcal{H}} \omega_j^i + \sum_{f \in \mathcal{F}} y_j^{f*} \text{ for all } j \in \mathbb{N}.$$
2. For every firm $f \in \mathcal{F}$, y^{f*} maximizes profits, i.e.,
$$p^* \cdot y^{f*} \geq p^* \cdot y \text{ for all } y \in Y^f.$$
3. For every consumer $i \in \mathcal{H}$, x^{i*} maximizes utility, i.e.,
$$u^i\left(x^{i*}\right) \geq u^i(x) \text{ for all } x \text{ s.t. } x \in X^i \text{ and } p^* \cdot x \leq p^* \cdot x^{i*}.$$

Welfare Theorems VII

- Establish existence of competitive equilibrium with finite number of commodities and standard convexity assumptions is straightforward.
- With infinite number of commodities, somewhat more difficult and requires more sophisticated arguments.

Definition A feasible allocation $(\boldsymbol{x}, \boldsymbol{y})$ for economy $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \boldsymbol{u}, \boldsymbol{\omega}, \boldsymbol{Y}, \boldsymbol{X}, \boldsymbol{\theta})$ is *Pareto optimal* if there exists no other feasible allocation $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ such that $\hat{x}^i \in X^i$ for all $i \in \mathcal{H}$, $\hat{y}^f \in Y^f$ for all $f \in \mathcal{F}$,

$$\sum_{i \in \mathcal{H}} \hat{x}_j^i \leq \sum_{i \in \mathcal{H}} \omega_j^i + \sum_{f \in \mathcal{F}} \hat{y}_j^f \text{ for all } j \in \mathbb{N},$$

and

$$u^i\left(\hat{x}^i\right) \geq u^i\left(x^i\right) \text{ for all } i \in \mathcal{H}$$

with at least one strict inequality.

Welfare Theorems VIII

Definition Household $i \in \mathcal{H}$ is *locally non-satiated* if at each x^i, $u^i\left(x^i\right)$ is strictly increasing in at least one of its arguments at x^i and $u^i\left(x^i\right) < \infty$.

- Latter requirement already implied by the fact that $u^i : X^i \to \mathbb{R}$.

Theorem **(First Welfare Theorem I)** Suppose that $\left(\boldsymbol{x}^*, \boldsymbol{y}^*, p^*\right)$ is a competitive equilibrium of economy $\mathcal{E} \equiv \left(\mathcal{H}, \mathcal{F}, \boldsymbol{u}, \boldsymbol{\omega}, \boldsymbol{Y}, \boldsymbol{X}, \boldsymbol{\theta}\right)$ with $\mathcal{H}$ finite. Assume that all households are locally non-satiated. Then $\left(\boldsymbol{x}^*, \boldsymbol{y}^*\right)$ is Pareto optimal.

Proof of First Welfare Theorem I

- To obtain a contradiction, suppose that there exists a feasible $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ such that $u^i\left(\hat{x}^i\right) \geq u^i\left(x^i\right)$ for all $i \in \mathcal{H}$ and $u^i\left(\hat{x}^i\right) > u^i\left(x^i\right)$ for all $i \in \mathcal{H}'$, where $\mathcal{H}'$ is a non-empty subset of $\mathcal{H}$.
- Since $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ is a competitive equilibrium, it must be the case that for all $i \in \mathcal{H}$,

$$\begin{aligned} p^* \cdot \hat{x}^i &\geq p^* \cdot x^{i*} \\ &= p^* \cdot \left(\omega^i + \sum_{f \in \mathcal{F}} \theta_f^i y^{f*}\right) \end{aligned} \tag{14}$$

and for all $i \in \mathcal{H}'$,

$$p^* \cdot \hat{x}^i > p^* \cdot \left(\omega^i + \sum_{f \in \mathcal{F}} \theta_f^i y^{f*}\right). \tag{15}$$

Proof of First Welfare Theorem II

- Second inequality follows immediately in view of the fact that x^{i*} is the utility maximizing choice for household i, thus if $\hat{x}^i$ is strictly preferred, then it cannot be in the budget set.
- First inequality follows with a similar reasoning. Suppose that it did not hold.
- Then by the hypothesis of local-satiation, u^i must be strictly increasing in at least one of its arguments, let us say the j'th component of x.
- Then construct $\hat{x}^i(\varepsilon)$ such that $\hat{x}^i_j(\varepsilon) = \hat{x}^i_j$ and $\hat{x}^i_{j'}(\varepsilon) = \hat{x}^i_{j'} + \varepsilon$.
- For $\varepsilon \downarrow 0$, $\hat{x}^i(\varepsilon)$ is in household i's budget set and yields strictly greater utility than the original consumption bundle x^i, contradicting the hypothesis that household i was maximizing utility.
- Note local non-satiation implies that $u^i\left(x^i\right) < \infty$, and thus the right-hand sides of (14) and (15) are finite.

Proof of First Welfare Theorem III

- Now summing over (14) and (15), we have

$$\begin{aligned} p^* \cdot \sum_{i \in \mathcal{H}} \hat{x}^i &> p^* \cdot \sum_{i \in \mathcal{H}} \left(\omega^i + \sum_{f \in \mathcal{F}} \theta_f^i y^{f*} \right), \\ &= p^* \cdot \left(\sum_{i \in \mathcal{H}} \omega^i + \sum_{f \in \mathcal{F}} y^{f*} \right), \end{aligned} \tag{16}$$

- Second line uses the fact that the summations are finite, can change the order of summation, and that by definition of shares $\sum_{i \in \mathcal{H}} \theta_f^i = 1$ for all f.
- Finally, since $\boldsymbol{y}^*$ is profit-maximizing at prices p^*, we have that

$$p^* \cdot \sum_{f \in \mathcal{F}} y^{f*} \geq p^* \cdot \sum_{f \in \mathcal{F}} y^f \text{ for any } \left\{ y^f \right\}_{f \in \mathcal{F}} \text{ with } y^f \in Y^f \text{ for all } f \in \mathcal{F} \tag{17}$$

Proof of First Welfare Theorem IV

- However, by feasibility of $\hat{x}^i$ (Definition above, part 1), we have

$$\sum_{i \in \mathcal{H}} \hat{x}_j^i \leq \sum_{i \in \mathcal{H}} \omega_j^i + \sum_{f \in \mathcal{F}} \hat{y}_j^f,$$

- Therefore, by multiplying both sides by p^* and exploiting (17),

$$\begin{aligned} p^* \cdot \sum_{i \in \mathcal{H}} \hat{x}_j^i & \leq p^* \cdot \left(\sum_{i \in \mathcal{H}} \omega_j^i + \sum_{f \in \mathcal{F}} \hat{y}_j^f \right) \\ & \leq p^* \cdot \left(\sum_{i \in \mathcal{H}} \omega_j^i + \sum_{f \in \mathcal{F}} y_j^{f*} \right), \end{aligned}$$

- Contradicts (16), establishing that any competitive equilibrium allocation $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is Pareto optimal.

Welfare Theorems IX

- Proof of the First Welfare Theorem based on two intuitive ideas.
  1. If another allocation Pareto dominates the competitive equilibrium, then it must be non-affordable in the competitive equilibrium.
  2. Profit-maximization implies that any competitive equilibrium already contains the maximal set of affordable allocations.
- Note it makes no convexity assumption.
- Also highlights the importance of the feature that the relevant sums exist and are finite.
  - Otherwise, the last step would lead to the conclusion that "$\infty < \infty$".
- That these sums exist followed from two assumptions: finiteness of the number of individuals and non-satiation.

Welfare Theorems X

Theorem **(First Welfare Theorem II)** Suppose that $(\boldsymbol{x}^*, \boldsymbol{y}^*, p^*)$ is a competitive equilibrium of the economy $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \boldsymbol{u}, \boldsymbol{\omega}, \boldsymbol{Y}, \boldsymbol{X}, \boldsymbol{\theta})$ with $\mathcal{H}$ countably infinite. Assume that all households are locally non-satiated and that $p^* \cdot \omega^* = \sum_{i\in\mathcal{H}} \sum_{j=0}^{\infty} p_j^* \omega_j^i < \infty$. Then $(\boldsymbol{x}^*, \boldsymbol{y}^*, p^*)$ is Pareto optimal.

- **Proof:**
  - Same as before but now local non-satiation does not guarantee summations are finite (16), since we sum over an infinite number of households.
  - But since endowments are finite, the assumption that $\sum_{i\in\mathcal{H}} \sum_{j=0}^{\infty} p_j^* \omega_j^i < \infty$ ensures that the sums in (16) are indeed finite.

Welfare Theorems X

- Second Welfare Theorem (converse to First): whether or not $\mathcal{H}$ is finite is not as important as for the First Welfare Theorem.
- But requires assumptions such as the convexity of consumption and production sets and preferences, and additional requirements because it contains an "existence of equilibrium argument".
- Recall that the consumption set of each individual $i \in \mathcal{H}$ is $X^i \subset \mathbb{R}_+^\infty$.
- A typical element of X^i is $x^i = \left(x_1^i, x_2^i, ...\right)$, where x_t^i can be interpreted as the vector of consumption of individual i at time t.
- Similarly, a typical element of the production set of firm $f \in \mathcal{F}$, Y^f, is $y^f = \left(y_1^f, y_2^f, ...\right)$.
- Let us define $x^i\left[T\right] = \left(x_0^i, x_1^i, x_2^i, ..., x_T^i, 0, 0, ...\right)$ and $y^f\left[T\right] = \left(y_0^f, y_1^f, y_2^f, ..., y_T^f, 0, 0, ...\right)$.
- It can be verified that $\lim_{T\to\infty} x^i\left[T\right] = x^i$ and $\lim_{T\to\infty} y^f\left[T\right] = y^f$ in the product topology.

Second Welfare Theorem I

Theorem

Consider a Pareto optimal allocation $(\boldsymbol{x}^{**}, \boldsymbol{y}^{**})$ *in an economy described by* $\boldsymbol{\omega}$, $\{Y^f\}_{f \in \mathcal{F}}$, $\{X^i\}_{i \in \mathcal{H}}$, *and* $\{u^i(\cdot)\}_{i \in \mathcal{H}}$. *Suppose all production and consumption sets are convex, all production sets are cones, and all* $\{u^i(\cdot)\}_{i \in \mathcal{H}}$ *are continuous and quasi-concave and satisfy local non-satiation. Suppose also that* $0 \in X^i$, *that for each* $x, x' \in X^i$ *with* $u^i(x) > u^i(x')$ *for all* $i \in \mathcal{H}$, *there exists* $\bar{T}$ *such that* $u^i(x[T]) > u^i(x')$ *for all* $T \geq \bar{T}$ *and for all* $i \in \mathcal{H}$, *and that for each* $y \in Y^f$, *there exists* $\tilde{T}$ *such that* $y[T] \in Y^f$ *for all* $T \geq \tilde{T}$ *and for all* $f \in \mathcal{F}$. *Then this allocation can be decentralized as a competitive equilibrium.*

Second Welfare Theorem II

Theorem

(continued) *In particular, there exist* p^{**} *and* $(\boldsymbol{\omega}^{**}, \boldsymbol{\theta}^{**})$ *such that*

1. $\boldsymbol{\omega}^{**}$ *satisfies* $\boldsymbol{\omega} = \sum_{i \in \mathcal{H}} \omega^{i**}$;
2. *for all* $f \in \mathcal{F}$,
$$p^{**} \cdot y^{f**} \geq p^{**} \cdot y \text{ for all } y \in Y^f;$$
3. *for all* $i \in \mathcal{H}$,
$$\text{if } x^i \in X^i \text{ involves } u^i\left(x^i\right) > u^i\left(x^{i**}\right), \text{ then } p^{**} \cdot x^i \geq p^{**} \cdot w^{i**},$$
where $w^{i**} \equiv \omega^{i**} + \sum_{f \in \mathcal{F}} \theta_f^{i**} y^{f**}$.

Moreover, if $p^{**} \cdot \mathbf{w}^{**} > 0$ *[i.e.,* $p^{**} \cdot w^{i**} > 0$ *for each* $i \in \mathcal{H}$*], then economy* $\mathcal{E}$ *has a competitive equilibrium* $(\mathbf{x}^{**}, \mathbf{y}^{**}, p^{**})$.

Welfare Theorems XII

- Notice:
  - if instead we had a finite commodity space, say with K commodities, then the hypothesis that $0 \in X^i$ for each $i \in \mathcal{H}$ and $x, x' \in X^i$ with $u^i(x) > u^i(x')$, there exists $\bar{T}$ such that $u^i(x[T]) > u^i(x'[T])$ for all $T \geq \bar{T}$ and all $i \in \mathcal{H}$ (and also that there exists $\tilde{T}$ such that if $y \in Y^f$, then $y[T] \in Y^f$ for all $T \geq \tilde{T}$ and all $f \in \mathcal{F}$) would be satisfied automatically, by taking $\bar{T} = \tilde{T} = K$.
  - Condition not imposed in Second Welfare Theorem in economies with a finite number of commodities.
  - In dynamic economies, its role is to ensure that changes in allocations at very far in the future should not have a large effect.
- The conditions for the Second Welfare Theorem are more difficult to satisfy than those for the First.
- Also the more important of the two theorems: stronger results that any Pareto optimal allocation can be *decentralized*.

Welfare Theorems XIII

- Immediate corollary is an existence result: a competitive equilibrium must exist.
- Motivates many to look for the set of Pareto optimal allocations instead of explicitly characterizing competitive equilibria.
- Real power of the Theorem in dynamic macro models comes when we combine it with models that admit a representative household.
- Enables us to characterize *the optimal growth allocation* that maximizes the utility of the representative household and assert that this will correspond to a competitive equilibrium.

Sketch of the Proof of SWT I

- First, I establish that there exists a price vector p^{**} and an endowment and share allocation $(\boldsymbol{\omega}^{**}, \boldsymbol{\theta}^{**})$ that satisfy conditions 1-3.
- This has two parts.
- (Part 1) This part follows from the Geometric Hahn-Banach Theorem.
- Define the "more preferred" sets for each $i \in \mathcal{H}$:

$$P^i = \left\{x^i \in X^i : u^i\left(x^i\right) > u^i\left(x^{i**}\right)\right\}.$$

- Clearly, each P^i is convex.
- Let $P = \sum_{i \in \mathcal{H}} P^i$ and $Y' = \sum_{f \in \mathcal{F}} Y^f + \{\boldsymbol{\omega}\}$, where recall that $\boldsymbol{\omega} = \sum_{i \in \mathcal{H}} \omega^{i**}$, so that Y' is the sum of the production sets shifted by the endowment vector.
- Both P and Y' are convex (since each P^i and each Y^f are convex).

Sketch of the Proof of SWT II

- Consider the sequences of production plans for each firm to be subsets of ℓ_∞^K, i.e., vectors of the form $y^f = \left(y_0^f, y_1^f, ...\right)$, with each $y_j^f \in \mathbb{R}_+^K$.
- Moreover, since each production set is a cone, $Y' = \sum_{f\in\mathcal{F}} Y^f + \{\boldsymbol{\omega}\}$ has an interior point.
- Moreover, let $x^{**} = \sum_{i\in\mathcal{H}} x^{i**}$.
- By feasibility and local non-satiation, $x^{**} = \sum_{f\in\mathcal{F}} y^{i**} + \boldsymbol{\omega}$.
- Then $x^{**} \in Y'$ and also $x^{**} \in \overline{P}$ (where $\overline{P}$ is the closure of P).
- Next, observe that $P \cap Y' = \varnothing$. Otherwise, there would exist $\tilde{y} \in Y'$, which is also in P.
- This implies that if distributed appropriately across the households, $\tilde{y}$ would make all households equally well off and at least one of them would be strictly better off

Sketch of the Proof of SWT III

- I.e., by the definition of the set P, there would exist $\left\{\tilde{x}^i\right\}_{i\in\mathcal{H}}$ such that $\sum_{i\in\mathcal{H}} \tilde{x}^i = \tilde{y}$, $\tilde{x}^i \in X^i$, and $u^i\left(\tilde{x}^i\right) \geq u^i\left(x^{i**}\right)$ for all $i \in \mathcal{H}$ with at least one strict inequality.
- This would contradict the hypothesis that $\left(x^{**}, y^{**}\right)$ is a Pareto optimum.
- Since Y' has an interior point, P and Y' are convex, and $P \cap Y' = \varnothing$, Geometric Theorem implies that there exists a nonzero continuous linear functional ϕ such that

$$\phi\left(y\right) \leq \phi\left(x^{**}\right) \leq \phi\left(x\right) \text{ for all } y \in Y' \text{ and all } x \in P. \tag{18}$$

- (Part 2) We next need to show that this linear functional can be interpreted as a price vector (i.e., that it does have an inner product representation).
- Let, $\bar{\phi}\left(x\right) = \lim_{T\to\infty} \phi\left(x\left[T\right]\right)$.

Sketch of the Proof of SWT IV

- Then, first note that if $\phi(x)$ is a continuous linear functional, then $\bar{\phi}(x) = \sum_{j=0}^{\infty} \bar{\phi}_j(x_j)$ is also a linear functional, where each $\bar{\phi}_j(x_j)$ is a linear functional on $X_j \subset \mathbb{R}_+^K$.
- Second claim follows from the fact that $\phi(x[T])$ is bounded above by $\|\phi\| \cdot \|x\|$, where $\|\phi\|$ denotes the norm of the functional ϕ and is thus finite.
- Clearly, $\|x\|$ is also finite.
- Moreover, since each element of x is nonnegative, $\{\phi(x[t])\}$ is a monotone sequence, thus $\lim_{T\to\infty} \phi(x[T])$ converges and we denote the limit by $\bar{\phi}(x)$.
- Moreover, this limit is a bounded functional and therefore from Continuity of Linear Function Theorem, it is continuous.

Sketch of the Proof of SWT V

- The first claim follows from the fact that since $x_j \in X_j \subset \mathbb{R}_+^K$, we can define a continuous linear functional on the dual of X_j by $\bar{\phi}_j\left(x_j\right) = \phi\left(\bar{x}^j\right) = \sum_{s=1}^K p_{j,s}^{**} x_{j,s}$, where $\bar{x}^j = \left(0, 0, ..., x_j, 0, ...\right)$ [i.e., $\bar{x}^j$ has x_j as jth element and zeros everywhere else].
- Then clearly,

$$\bar{\phi}\left(x\right) = \sum_{j=0}^{\infty} \bar{\phi}_j\left(x_j\right) = \sum_{s=0}^{\infty} p_s^{**} x_s = p^{**} \cdot x.$$

- To complete this part of the proof, we only need to show that $\bar{\phi}\left(x\right) = \sum_{j=0}^{\infty} \bar{\phi}_j\left(x_j\right)$ can be used instead of ϕ as the continuous linear functional in (18).

Sketch of the Proof of SWT VI

- This follows immediately from the hypothesis that $0 \in X^i$ for each $i \in \mathcal{H}$ and that there exists $\bar{T}$ such that for any $x, x' \in X^i$ with $u^i(x) > u^i(x')$, $u^i(x[T]) > u^i(x'[T])$ for all $T \geq \bar{T}$ and for all $i \in \mathcal{H}$, and that there exists $\tilde{T}$ such that if $y \in Y^f$, then $y[T] \in Y^f$ for all $T \geq \tilde{T}$ and for all $f \in \mathcal{F}$.
- In particular, take $T' = \max\left\{\bar{T}, \tilde{T}\right\}$ and fix $x \in P$.
- Since x has the property that $u^i\left(x^i\right) > u^i\left(x^{i**}\right)$ for all $i \in \mathcal{H}$, we also have that $u^i\left(x^i[T]\right) > u^i\left(x^{i**}[T]\right)$ for all $i \in \mathcal{H}$ and $T \geq T'$.
- Therefore,

$$\phi\left(x^{**}[T]\right) \leq \phi\left(x[T]\right) \text{ for all } x \in P.$$

- Now taking limits,

$$\bar{\phi}\left(x^{**}\right) \leq \bar{\phi}(x) \text{ for all } x \in P.$$

Sketch of the Proof of SWT VII

- A similar argument establishes that $\bar{\phi}(x^{**}) \geq \bar{\phi}(y)$ for all $y \in Y'$, so that $\bar{\phi}(x)$ can be used as the continuous linear functional separating P and Y'.
- Since $\bar{\phi}_j(x_j)$ is a linear functional on $X_j \subset \mathbb{R}_+^K$, it has an inner product representation, $\bar{\phi}_j(x_j) = p_j^{**} \cdot x_j$ and therefore so does $\bar{\phi}(x) = \sum_{j=0}^{\infty} \bar{\phi}_j(x_j) = p^{**} \cdot x$.
- Parts 1 and 2 have therefore established that there exists a price vector (functional) p^{**} such that conditions 2 and 3 hold.
- Condition 1 is satisfied by construction.
- Condition 2 is sufficient to establish that all firms maximize profits at the price vector p^{**}.
- To show that all consumers maximize utility at the price vector p^{**}, use the hypothesis that $p^{**} \cdot w^{i**} > 0$ for each $i \in \mathcal{H}$.

Sketch of the Proof of SWT VIII

- We know from Condition 3 that if $x^i \in X^i$ involves $u^i\left(x^i\right) > u^i\left(x^{i**}\right)$, then $p^{**} \cdot x^i \geq p^{**} \cdot w^{i**}$.
- This implies that if there exists x^i that is strictly preferred to x^{i**} and satisfies $p^{**} \cdot x^i = p^{**} \cdot w^{i**}$ (which would amount to the consumer not maximizing utility), then there exists $x^i - \varepsilon$ for ε small enough, such that $u^i\left(x^i - \varepsilon\right) > u^i\left(x^{i**}\right)$, then $p^{**} \cdot \left(x^i - \varepsilon\right) < p^{**} \cdot w^{i**}$, thus violating Condition 3.
- Therefore, consumers also maximize utility at the price p^{**}, establishing that $\left(\boldsymbol{x}^{**}, \boldsymbol{y}^{**},\ p^{**}\right)$ is a competitive equilibrium. $\square$

Sequential Trading I

- Standard general equilibrium models assume all commodities are traded at a given point in time—and once and for all.
- When trading same good in different time periods or states of nature, trading once and for all less reasonable.
- In models of economic growth, typically assume trading takes place at different points in time.
- But with complete markets, sequential trading gives the same result as trading at a single point in time.
- *Arrow-Debreu equilibrium* of dynamic general equilibrium model: all households trading at $t = 0$ and purchasing and selling irrevocable claims to commodities indexed by date and state of nature.
- Sequential trading: separate markets at each t, households trading labor, capital and consumption goods in each such market.
- With complete markets (and time consistent preferences), both are equivalent.

Sequential Trading II

- *(Basic) Arrow Securities*: means of transferring resources across different dates and different states of nature.
- Households can trade Arrow securities and then use these securities to purchase goods at different dates or after different states of nature.
- Reason why both are equivalent:
  - by definition of competitive equilibrium, households correctly anticipate all the prices and purchase sufficient Arrow securities to cover the expenses that they will incur.
- Instead of buying claims at time $t = 0$ for $x^h_{i,t'}$ units of commodity $i = 1, ..., N$ at date t' at prices $(p_{1,t'}, ..., p_{N,t'})$, sufficient for household h to have an income of $\sum_{i=1}^{N} p_{i,t'} x^h_{i,t'}$ and know that it can purchase as many units of each commodity as it wishes at time t' at the price vector $(p_{1,t'}, ..., p_{N,t'})$.
- Consider a dynamic exchange economy running across periods $t = 0, 1, ..., T$, possibly with $T = \infty$.

Sequential Trading III

- There are N goods at each date, denoted by $(x_{1,t}, ..., x_{N,t})$.
- Let the consumption of good i by household h at time t be denoted by $x_{i,t}^h$.
- Goods are perishable, so that they are indeed consumed at time t.
- Each household $h \in \mathcal{H}$ has a vector of endowment $\left(\omega_{1,t}^h, ..., \omega_{N,t}^h\right)$ at time t, and preferences

$$\sum_{t=0}^{T} \beta_h^t u^h \left(x_{1,t}^h, ..., x_{N,t}^h\right),$$

  for some $\beta_h \in (0,1)$.
- These preferences imply no externalities and are time consistent.
- All markets are open and competitive.
- Let an Arrow-Debreu equilibrium be given by $(\boldsymbol{p}^*, \boldsymbol{x}^*)$, where $\boldsymbol{x}^*$ is the complete list of consumption vectors of each household $h \in \mathcal{H}$.

Sequential Trading IV

- That is,
$$\boldsymbol{x}^* = \left(x_{1,0}, ...x_{N,0}, ..., x_{1,T}, ...x_{N,T}\right),$$
with $x_{i,t} = \left\{x_{i,t}^h\right\}_{h\in\mathcal{H}}$ for each i and t.
- $\boldsymbol{p}^*$ is the vector of complete prices $\boldsymbol{p}^* = \left(p_{1,0}^*, ..., p_{N,0}^*, ..., p_{1,T}^*, ..., p_{N,T}^*\right)$, with $p_{1,0}^* = 1$.
- Arrow-Debreu equilibrium: trading only at $t = 0$ and choose allocation that satisfies
$$\sum_{t=0}^{T}\sum_{i=1}^{N} p_{i,t}^* x_{i,t}^h \leq \sum_{t=0}^{T}\sum_{i=1}^{N} p_{i,t}^* \omega_{i,t}^h \text{ for each } h \in \mathcal{H}.$$
- Market clearing then requires
$$\sum_{h\in\mathcal{H}} x_{i,t}^h \leq \sum_{h\in\mathcal{H}} \omega_{i,t}^h \text{ for each } i = 1, ..., N \text{ and } t = 0, 1, ..., T.$$

Sequential Trading V

- Equilibrium with sequential trading:
  - Markets for goods dated t open at time t.
  - There are T bonds—*Arrow securities*—in zero net supply that can be traded at $t = 0$.
  - Bond indexed by t pays one unit of one of the goods, say good $i = 1$ at time t.
- Prices of bonds denoted by $(q_1, ..., q_T)$, expressed in units of good $i = 1$ (at time $t = 0$).
- Thus a household can purchase a unit of bond t at time 0 by paying q_t units of good 1 and will receive one unit of good 1 at time t
- Denote purchase of bond t by household h by $b_t^h \in \mathbb{R}$.
- Since each bond is in zero net supply, market clearing requires

$$\sum_{h \in \mathcal{H}} b_t^h = 0 \text{ for each } t = 0, 1, ..., T.$$

Sequential Trading VI

- Each individual uses his endowment plus (or minus) the proceeds from the corresponding bonds at each date t.
- Convenient (and possible) to choose a separate numeraire for each date t, $p^{**}_{1,t} = 1$ for all t.
- Therefore, the budget constraint of household $h \in \mathcal{H}$ at time t, given equilibrium $(\boldsymbol{p}^{**}, \boldsymbol{q}^{**})$:

$$\sum_{i=1}^{N} p^{**}_{i,t} x^h_{i,t} \leq \sum_{i=1}^{N} p^{**}_{i,t} \omega^h_{i,t} + b^h_t \text{ for } t = 0, 1, ..., T, \tag{19}$$

together with the constraint

$$\sum_{t=0}^{T} q^{**}_t b^h_t \leq 0$$

with the normalization that $q^{**}_0 = 1$.

Sequential Trading VII

- Let equilibrium with sequential trading be $(\boldsymbol{p}^{**}, \boldsymbol{q}^{**}, \boldsymbol{x}^{**}, \boldsymbol{b}^{**})$.

Theorem **(Sequential Trading)** For the above-described economy, if $(\boldsymbol{p}^{*}, \boldsymbol{x}^{*})$ is an Arrow-Debreu equilibrium, then there exists a sequential trading equilibrium $(\boldsymbol{p}^{**}, \boldsymbol{q}^{**}, \boldsymbol{x}^{**}, \boldsymbol{b}^{**})$, such that $\boldsymbol{x}^{*} = \boldsymbol{x}^{**}$, $p^{**}_{i,t} = p^{*}_{i,t} / p^{*}_{1,t}$ for all i and t and $q^{**}_{t} = p^{*}_{1,t}$ for all $t > 0$. Conversely, if $(\boldsymbol{p}^{**}, \boldsymbol{q}^{**}, \boldsymbol{x}^{**}, \boldsymbol{b}^{**})$ is a sequential trading equilibrium, then there exists an Arrow-Debreu equilibrium $(\boldsymbol{p}^{*}, \boldsymbol{x}^{*})$ with $\boldsymbol{x}^{*} = \boldsymbol{x}^{**}$, $p^{*}_{i,t} = p^{**}_{i,t} p^{*}_{1,t}$ for all i and t, and $p^{*}_{1,t} = q^{**}_{t}$ for all $t > 0$.

- Focus on economies with sequential trading and assume that there exist Arrow securities to transfer resources across dates.
- These securities might be riskless bonds in zero net supply, or without uncertainty, role typically played by the capital stock.
- Also typically normalize the price of one good at each date to 1.
- Hence interest rates are key relative prices in dynamic models.

Optimal Growth in Discrete Time I

- Economy characterized by an aggregate production function, and a representative household.
- Optimal growth problem in discrete time with no uncertainty, no population growth and no technological progress:

$$\max_{\{c_t, k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{20}$$

subject to

$$k_{t+1} = f(k_t) + (1-\delta) k_t - c_t, \tag{21}$$

$k_t \geq 0$ and given $k_0 > 0$.

- Initial level of capital stock is k_0, but this gives a single initial condition.

Optimal Growth in Discrete Time II

- Solution will correspond to two difference equations, thus need another boundary condition
- Will come from the optimality of a dynamic plan in the form of a *transversality condition*.
- Can be solved in a number of different ways: e.g., infinite dimensional Lagrangian, but the most convenient is by *dynamic programming.*
- Note even if we wished to bypass the Second Welfare Theorem and directly solve for competitive equilibria, we would have to solve a problem similar to the maximization of (20) subject to (21).

Optimal Growth in Discrete Time III

- Assuming that the representative household has one unit of labor supplied inelastically, this problem can be written as:

$$\max_{\{c_t, k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to some given a_0 and

$$a_{t+1} = R_t \left[a_t - c_t + w_t \right], \tag{22}$$

- Need an additional condition so that this flow budget constraint eventually converges (i.e., so that a_t should not go to negative infinity).
- Can impose a lifetime budget constraint, or augment flow budget constraint with another condition to rule out wealth going to negative infinity.

Optimal Growth in Continuous Time

- The formulation of the optimal growth problem in continuous time is very similar:

$$\max_{[c(t),k(t)]_{t=0}^{\infty}} \int_0^{\infty} \exp\left(-\rho t\right) u\left(c\left(t\right)\right) dt \tag{23}$$

subject to

$$\dot{k}\left(t\right) = f\left(k\left(t\right)\right) - c\left(t\right) - \delta k\left(t\right), \tag{24}$$

$k\left(t\right) \geq 0$ and given $k\left(0\right) = k_0 > 0$.

- The objective function (23) is the direct continuous-time analog of (20), and (24) gives the resource constraint of the economy, similar to (21) in discrete time.
- Again, lacks one boundary condition which will come from the transversality condition.
- Most convenient way of characterizing the solution to this problem is via *optimal control theory.*

Conclusions

- Models we study in this book are examples of more general dynamic general equilibrium models.
- First and the Second Welfare Theorems are essential.
- The most general class of dynamic general equilibrium models are not tractable enough to derive sharp results about economic growth.
- Need simplifying assumptions, the most important one being the representative household assumption.