Foundations of Neoclassical Growth

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Macroeconomics II
Foundations of Neoclassical Growth

- Solow model: constant saving rate.
- More satisfactory to specify the *preference orderings* of individuals and derive their decisions from these preferences.
- Enables better understanding of the factors that affect savings decisions.
- Enables to discuss the “optimality” of equilibria.
- Whether the (competitive) equilibria of growth models can be “improved upon”.
- Notion of improvement: Pareto optimality.
Consider an economy consisting of a unit measure of infinitely-lived households.

I.e., an uncountable number of households: e.g., the set of households \( \mathcal{H} \) could be represented by the unit interval \([0, 1]\).

Emphasize that each household is infinitesimal and will have no effect on aggregates.

Can alternatively think of \( \mathcal{H} \) as a countable set of the form \( \mathcal{H} = \{1, 2, \ldots, M\} \) with \( M = \infty \), without any loss of generality.

Advantage of unit measure: averages and aggregates are the same.

Simpler to have \( \mathcal{H} \) as a finite set in the form \( \{1, 2, \ldots, M\} \) with \( M \) large but finite.

Acceptable for many models, but with overlapping generations require the set of households to be infinite.
How to model households in infinite horizon?

1. “infinitely lived” or consisting of overlapping generations with full altruism linking generations → infinite planning horizon
2. Overlapping generations → finite planning horizon (generally...).
Time Separable Preferences

- Standard assumptions on preference orderings so that they can be represented by utility functions.
- In particular, each household $i$ has an *instantaneous utility function*
  \[ u_i(c_{it}), \]
  \[ u_i : \mathbb{R}_+ \rightarrow \mathbb{R} \] is increasing and concave and $c_{it}$ is the consumption of household $i$ in period $t$.
- Note instantaneous utility function is *not* specifying a complete preference ordering over all commodities—here consumption levels in all dates.
- Sometimes also referred to as the “felicity function”.
- Two major assumptions in writing an instantaneous utility function
  1. consumption externalities are ruled out.
  2. overall utility is *time separable*.
Start with the case of infinite planning horizon.

Suppose households discount the future “exponentially”—or “proportionally”.

Interpret $u_i(\cdot)$ as a “Bernoulli utility function”.

Then preferences of household $i$ at time $t = 0$ can be represented by a von Neumann-Morgenstern expected utility function.

Thus household preferences at time $t = 0$ are

$$E_0^i \sum_{t=0}^{\infty} \beta_i^t u_i(c_{it}), \quad (1)$$

where $\beta_i \in (0, 1)$ is the discount factor of household $i$. 

Heterogeneity and the Representative Household

- $E^i_0$ is the expectation operator with respect to the information set available to household $i$ at time $t = 0$.
- So far index individual utility function, $u_i(\cdot)$, and the discount factor, $\beta_i$, by “$i$”.
- Households could also differ according to their income processes. E.g., effective labor endowments of $\{e_{it}\}_{t=0}^{\infty}$, labor income of $\{e_{it}w_t\}_{t=0}^{\infty}$.
- But at this level of generality, this problem is not tractable.
- Follow the standard approach in macroeconomics and assume the existence of a representative household.
Exponential discounting and time separability: ensure “time-consistent” behavior.

A solution \( \{ x_t \}_{t=0}^T \) (possibly with \( T = \infty \)) is time consistent if:
- whenever \( \{ x_t \}_{t=0}^T \) is an optimal solution starting at time \( t = 0 \),
  \( \{ x_t \}_{t=t'}^T \) is an optimal solution to the continuation dynamic optimization problem starting from time \( t = t' \in [0, T] \).
An economy admits a representative household if preference side can be represented as if a single household made the aggregate consumption and saving decisions subject to a single budget constraint.

This description concerning a representative household is purely positive.

Stronger notion of “normative” representative household: if we can also use the utility function of the representative household for welfare comparisons.

Simplest case that will lead to the existence of a representative household: suppose each household is identical.
I.e., same $\beta$, same sequence $\{e_t\}_{t=0}^{\infty}$ and same $u(c_{it})$

where $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is increasing and concave and $c_{it}$ is the consumption of household $i$.

Again ignoring uncertainty, preference side can be represented as the solution to

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t),$$

(2)

$\beta \in (0, 1)$ is the common discount factor and $c_t$ the consumption level of the representative household.

Admits a representative household rather trivially.

Representative household’s preferences, (2), can be used for positive and normative analysis.
Representative Household III

- If instead households are not identical but assume can model as if demand side generated by the optimization decision of a representative household...
- More realistic, but:
  1. The representative household will have positive, but not always a normative meaning.
  2. Models with heterogeneity: often do not lead to behavior that can be represented as if generated by a representative household.

**Theorem** *(Debreu-Mantel-Sonnenschein Theorem)* Let $\varepsilon > 0$ be a scalar and $N < \infty$ be a positive integer. Consider a set of prices $P_\varepsilon = \{p \in \mathbb{R}_+^N : p_j / p_{j'} \geq \varepsilon \text{ for all } j \text{ and } j'\}$ and any continuous function $x : P_\varepsilon \to \mathbb{R}_+^N$ that satisfies Walras’ Law and is homogeneous of degree 0. Then there exists an exchange economy with $N$ commodities and $H < \infty$ households, where the aggregate demand is given by $x(p)$ over the set $P_\varepsilon$. 
That excess demands come from optimizing behavior of households puts no restrictions on the form of these demands.

- E.g., \( x(p) \) does not necessarily possess a negative-semi-definite Jacobian or satisfy the weak axiom of revealed preference (requirements of demands generated by individual households).

Hence without imposing further structure, impossible to derive specific \( x(p) \)'s from the maximization behavior of a single household.

Severe warning against the use of the representative household assumption.

Partly an outcome of very strong income effects:

- special but approximately realistic preference functions, and restrictions on distribution of income rule out arbitrary aggregate excess demand functions.
Gorman Aggregation

- Recall an indirect utility function for household $i$, $v_i(p, y^i)$, specifies (ordinal) utility as a function of the price vector $p = (p_1, ..., p_N)$ and household’s income $y^i$.
- $v_i(p, y^i)$: homogeneous of degree 0 in $p$ and $y$.

**Theorem (Gorman’s Aggregation Theorem)** Consider an economy with a finite number $N < \infty$ of commodities and a set $\mathcal{H}$ of households. Suppose that the preferences of household $i \in \mathcal{H}$ can be represented by an indirect utility function of the form

$$v^i(p, y^i) = a^i(p) + b(p)y^i,$$ (3)

then these preferences can be aggregated and represented by those of a representative household, with indirect utility

$$v(p, y) = \int_{i \in \mathcal{H}} a^i(p) \, di + b(p)y,$$

where $y \equiv \int_{i \in \mathcal{H}} y^i \, di$ is aggregate income.
Linear Engel Curves

- Demand for good $j$ (from Roy's identity):

$$x_j^i (p, y^i) = - \frac{1}{b(p)} \frac{\partial a^i(p)}{\partial p_j} - \frac{1}{b(p)} \frac{\partial b(p)}{\partial p_j} y^i.$$

- Thus linear Engel curves.

- “Indispensable” for the existence of a representative household.

- Let us say that there exists a *strong representative household* if redistribution of income or endowments across households does not affect the demand side.

- Gorman preferences are sufficient for a strong representative household.

- Moreover, they are also *necessary* (with the same $b(p)$ for all households) for the economy to admit a strong representative household.

- The proof is easy by a simple variation argument.
Importance of Gorman Preferences

- Gorman Preferences limit the **extent of income effects** and enables the aggregation of individual behavior.
- Integral is “Lebesgue integral,” so when $\mathcal{H}$ is a finite or countable set, $\int_{i \in \mathcal{H}} y^i \, di$ is indeed equivalent to the summation $\sum_{i \in \mathcal{H}} y^i$.
- Stated for an economy with a finite number of commodities, but can be generalized for infinite or even a continuum of commodities.
- Note all we require is there exists a monotonic transformation of the indirect utility function that takes the form in (3)—as long as no uncertainty.
- Contains some commonly-used preferences in macroeconomics.
Example: Constant Elasticity of Substitution Preferences

- A very common class of preferences: constant elasticity of substitution (CES) preferences or Dixit-Stiglitz preferences.
- Suppose each household denoted by $i \in \mathcal{H}$ has total income $y^i$ and preferences defined over $j = 1, \ldots, N$ goods

\[
U^i (x^i_1, \ldots, x^i_N) = \left[ \sum_{j=1}^{N} \left( x^i_j - \xi^i_j \right) \frac{\sigma-1}{\sigma} \right]^{\frac{\sigma}{\sigma-1}},
\]

where $\sigma \in (0, \infty)$ and $\xi^i_j \in [-\bar{\xi}, \bar{\xi}]$ is a household specific term, which parameterizes whether the particular good is a necessity for the household.

- For example, $\xi^i_j > 0$ may mean that household $i$ needs to consume a certain amount of good $j$ to survive.
Example II

- If we define the level of consumption of each good as $\hat{x}_j^i = x_j^i - \xi_j^i$, the elasticity of substitution between any two $\hat{x}_j^i$ and $\hat{x}_j^i'$ would be equal to $\sigma$.

- Each consumer faces a vector of prices $p = (p_1, ..., p_N)$, and we assume that for all $i$,

$$\sum_{j=1}^{N} p_j \xi^i < y^i,$$

- Thus household can afford a bundle such that $\hat{x}_j^i \geq 0$ for all $j$.

- The indirect utility function is given by

$$v^i (p, y^i) = \frac{\left[ - \sum_{j=1}^{N} p_j \xi^i + y^i \right]}{\left[ \sum_{j=1}^{N} p_j^{1-\sigma} \right]^{-\frac{1}{1-\sigma}}}, \quad (5)$$
Example III

- Satisfies the Gorman form (and is also homogeneous of degree 0 in $p$ and $y$).
- Therefore, this economy admits a representative household with indirect utility:

$$v(p, y) = \left[ -\sum_{j=1}^{N} p_j \xi_j + y \right] \left[ \sum_{j=1}^{N} p_j^{1-\sigma} \right]^{\frac{1}{1-\sigma}}$$

- $y$ is aggregate income given by $y \equiv \int_{i \in H} y^i \, di$ and $\xi_j \equiv \int_{i \in H} \xi^i \, di$.
- The utility function leading to this indirect utility function is

$$U(x_1, \ldots, x_N) = \left[ \sum_{j=1}^{N} (x_j - \xi_j)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}.$$ (6)

- Preferences closely related to CES preferences will be key in ensuring balanced growth in neoclassical growth models.
Gorman preferences also imply the existence of a normative representative household.

Recall an allocation is *Pareto optimal* if no household can be made strictly better-off without some other household being made worse-off.
Existence of Normative Representative Household

**Theorem (Existence of a Normative Representative Household)**
Consider an economy with a finite number $N < \infty$ of commodities, a set $\mathcal{H}$ of households and a convex aggregate production possibilities set $Y$. Suppose that the preferences of each household $i \in \mathcal{H}$ take the Gorman form, $v^i (p, y^i) = a^i (p) + b (p) y^i$.

1. Then any allocation that maximizes the utility of the representative household, $v (p, y) = \sum_{i \in \mathcal{H}} a^i (p) + b (p) y$, with $y \equiv \sum_{i \in \mathcal{H}} y^i$, is Pareto optimal.

2. Moreover, if $a^i (p) = a^i$ for all $p$ and all $i \in \mathcal{H}$, then any Pareto optimal allocation maximizes the utility of the representative household.
Proof of Theorem I

Represent a Pareto optimal allocation as:

$$\max_{\{p_j\}, \{y^i\}, \{z_j\}} \sum_{i \in \mathcal{H}} \alpha^i v^i (p, y^i) = \sum_{i \in \mathcal{H}} \alpha^i (a^i(p) + b(p)y^i)$$

subject to

$$-\frac{1}{b(p)} \left( \sum_{i \in \mathcal{H}} \frac{\partial a^i(p)}{\partial p_j} + \frac{\partial b(p)}{\partial p_j} y \right) = z_j \in Y_j(p) \text{ for } j = 1, ..., N$$

$$\sum_{i \in \mathcal{H}} y^i = y \equiv \sum_{j=1}^{N} p_j z_j$$

$$\sum_{j=1}^{N} p_j \omega_j = y,$$

$$p_j \geq 0 \text{ for all } j.$$
Proof of Theorem II

Here \( \{ \alpha^i \}_{i \in H} \) are nonnegative Pareto weights with \( \sum_{i \in H} \alpha^i = 1 \) and \( z_j \in Y_j(p) \) profit maximizing production of good \( j \).

First set of constraints use Roy’s identity to express total demand for good \( j \) and set it equal to supply, \( z_j \).

Second equation sets value of income equal to value of production.

Third equation makes sure total income is equal to the value of the endowments, \( \omega_j \).

Compare the above maximization problem to:

\[
\max \sum_{i \in H} a^i(p) + b(p)y
\]

subject to the same set of constraints.

The only difference is in the latter each household has been assigned the same weight.
Proof of Theorem III

- Let \((p^*, y^*)\) be a solution to the second problem.
- By definition it is also a solution to the first problem with \(\alpha^i = \alpha\), and therefore it is Pareto optimal.
- This establishes the first part of the theorem.
- To establish the second part, suppose that \(a^i(p) = a^i\) for all \(p\) and all \(i \in H\).
- To obtain a contradiction, let \(y \in \mathbb{R}^{|H|}\) and suppose that \((p_{\alpha^{**}}, y_{\alpha^{**}})\) is a solution to the first problem for some weights \(\{\alpha^i\}_{i \in H}\) and suppose that it is not a solution to the second problem.
- Let

  \[
  \alpha^M = \max_{i \in H} \alpha^i
  \]

  and

  \[
  H^M = \{ i \in H \mid \alpha^i = \alpha^M \}
  \]

  be the set of households given the maximum Pareto weight.
Proof of Theorem IV

- Let \((p^*, y^*)\) be a solution to the second problem such that

\[ y^i = 0 \text{ for all } i \notin \mathcal{H}^M. \]  \hspace{1cm} (7)

- Such a solution exists since objective function and constraint set in the second problem depend only on the vector \((y^1, \ldots, y^{\mid\mathcal{H}\mid})\) through

\[ y = \sum_{i \in \mathcal{H}} y^i. \]

- Since, by definition, \((p^{**}, y^{**})\) is in the constraint set of the second problem and is not a solution,

\[
\sum_{i \in \mathcal{H}} a^i + b(p^*) y^* > \sum_{i \in \mathcal{H}} a^i + b(p^{**}) y^{**} \hspace{1cm} (8)
\]

\[
b(p^*) y^* > b(p^{**}) y^{**}.
\]
Proof of Theorem V

The hypothesis that it is a solution to the first problem also implies

\[ \sum_{i \in H} \alpha^i a^i + \sum_{i \in H} \alpha^i b\left(p^*_\alpha\right)\left(y^*_\alpha\right)^i \geq \sum_{i \in H} \alpha^i a^i + \sum_{i \in H} \alpha^i b\left(p^*\right)\left(y^*\right)^i \]

\[ \sum_{i \in H} \alpha^i b\left(p^*_\alpha\right)\left(y^*_\alpha\right)^i \geq \sum_{i \in H} \alpha^i b\left(p^*\right)\left(y^*\right)^i. \]  \(9\)

Then, it can be seen that any solution \((p^{**}, y^{**})\) to the Pareto optimal allocation problem satisfies \(y^i = 0\) for any \(i \notin H^M\).

In view of this and the choice of \((p^*, y^*)\) in (7), equation (9) implies

\[ \alpha^M b\left(p^*_\alpha\right) \sum_{i \in H} \left(y^*_\alpha\right)^i \geq \alpha^M b\left(p^*\right) \sum_{i \in H} \left(y^*\right)^i \]

\[ b\left(p^*_\alpha\right)\left(y^*_\alpha\right) \geq b\left(p^*\right)\left(y^*\right), \]

Contradicts equation (8): hence under the stated assumptions, any Pareto optimal allocation maximizes the utility of the representative household.
Infinite Planning Horizon I

- Most growth and macro models assume that individuals have an infinite-planning horizon
- Two reasonable microfoundations for this assumption
  - First: “Poisson death model” or the *perpetual youth model*: individuals are finitely-lived, but not aware of when they will die.
    - 1. Strong simplifying assumption: likelihood of survival to the next age in reality is not a constant
    - 2. But a good starting point, tractable and implies expected lifespan of $1/\nu < \infty$ periods, can be used to get a sense value of $\nu$.
- Suppose each individual has a standard instantaneous utility function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$, and a “true” or “pure” discount factor $\hat{\beta}$
- Normalize $u(0) = 0$ to be the utility of death.
- Consider an individual who plans to have a consumption sequence $\{c_t\}_{t=0}^\infty$ (conditional on living).
Individual would have an expected utility at time $t = 0$ given by

$$U(0) = u(c_0) + \hat{\beta}(1 - \nu)u(c_1) + \hat{\beta}\nu u(0) + \hat{\beta}^2 (1 - \nu)^2 u(c_2) + \hat{\beta}^2 (1 - \nu)\nu u(0) + \ldots$$

$$= \sum_{t=0}^{\infty} \left( \hat{\beta}(1 - \nu) \right)^t u(c_t)$$

$$= \sum_{t=0}^{\infty} \beta^t u(c_t), \quad (10)$$

Second line collects terms and uses $u(0) = 0$, third line defines $\beta \equiv \hat{\beta}(1 - \nu)$ as “effective discount factor.”

Isomorphic to model of infinitely-lived individuals, but values of $\beta$ may differ.

Also equation (10) is already the expected utility; probabilities have been substituted.
Second: intergenerational altruism or from the “bequest” motive.

Imagine an individual who lives for one period and has a single offspring (who will also live for a single period and beget a single offspring etc.).

Individual not only derives utility from his consumption but also from the bequest he leaves to his offspring.

For example, utility of an individual living at time $t$ is given by

$$u(c_t) + U^b(b_t),$$

$c_t$ is his consumption and $b_t$ denotes the bequest left to his offspring.

For concreteness, suppose that the individual has total income $y_t$, so that his budget constraint is

$$c_t + b_t \leq y_t.$$
Infinite Planning Horizon IV

- $U^b (\cdot)$: how much the individual values bequests left to his offspring.
- Benchmark might be “purely altruistic:” cares about the utility of his offspring (with some discount factor).
- Let discount factor between generations be $\beta$.
- Assume offspring will have an income of $w$ without the bequest.
- Then the utility of the individual can be written as

$$u (c_t) + \beta V (b_t + w),$$

- $V (\cdot)$: continuation value, the utility that the offspring will obtain from receiving a bequest of $b_t$ (plus his own $w$).
- Value of the individual at time $t$ can in turn be written as

$$V (y_t) = \max_{c_t + b_t \leq y_t} \{u (c_t) + \beta V (b_t + w_{t+1})\},$$
Infinite Planning Horizon

Infinite Horizon

Canonical form of a dynamic programming representation of an infinite-horizon maximization problem.

Under some mild technical assumptions, this dynamic programming representation is equivalent to maximizing

$$\sum_{s=0}^{\infty} \beta^s u(c_{t+s})$$

at time $t$.

Each individual internalizes utility of all future members of the “dynasty”.

Fully altruistic behavior within a dynasty (“dynastic” preferences) will also lead to infinite planning horizon.
The Representative Firm I

- While not all economies would admit a representative household, standard assumptions (in particular no production externalities and competitive markets) are sufficient to ensure a representative firm.

**Theorem** *(The Representative Firm Theorem)* Consider a competitive production economy with $N \in \mathbb{N} \cup \{+\infty\}$ commodities and a countable set $\mathcal{F}$ of firms, each with a convex production possibilities set $Y^f \subset \mathbb{R}^N$. Let $p \in \mathbb{R}_+^N$ be the price vector in this economy and denote the set of profit maximizing net supplies of firm $f \in \mathcal{F}$ by $\hat{Y}^f (p) \subset Y^f$ (so that for any $\hat{y}^f \in \hat{Y}^f (p)$, we have $p \cdot \hat{y}^f \geq p \cdot y^f$ for all $y^f \in Y^f$). Then there exists a *representative firm* with production possibilities set $Y \subset \mathbb{R}^N$ and set of profit maximizing net supplies $\hat{Y} (p)$ such that for any $p \in \mathbb{R}_+^N$, $\hat{y} \in \hat{Y} (p)$ if and only if $\hat{y} (p) = \sum_{f \in \mathcal{F}} \hat{y}^f$ for some $\hat{y}^f \in \hat{Y}^f (p)$ for each $f \in \mathcal{F}$. 
Proof of Theorem: The Representative Firm I

- Let $Y$ be defined as follows:

$$Y = \left\{ \sum_{f \in F} y^f : y^f \in Y^f \text{ for each } f \in F \right\}.$$ 

- To prove the “if” part of the theorem, fix $p \in \mathbb{R}_+^N$ and construct $\hat{y} = \sum_{f \in F} \hat{y}^f$ for some $\hat{y}^f \in \hat{Y}^f(p)$ for each $f \in F$.

- Suppose, to obtain a contradiction, that $\hat{y} \notin \hat{Y}(p)$, so that there exists $y'$ such that $p \cdot y' > p \cdot \hat{y}$. 
Proof of Theorem: The Representative Firm II

- By definition of the set $Y$, this implies that there exists $\{y^f\}_{f \in \mathcal{F}}$ with $y^f \in Y^f$ such that

$$
\begin{align*}
p \cdot \left( \sum_{f \in \mathcal{F}} y^f \right) &> p \cdot \left( \sum_{f \in \mathcal{F}} \hat{y}^f \right) \\
\sum_{f \in \mathcal{F}} p \cdot y^f &> \sum_{f \in \mathcal{F}} p \cdot \hat{y}^f,
\end{align*}
$$

so that there exists at least one $f' \in \mathcal{F}$ such that

$$
p \cdot y^{f'} > p \cdot \hat{y}^{f'},
$$

- Contradicts the hypothesis that $\hat{y}^f \in \hat{Y}^f(p)$ for each $f \in \mathcal{F}$ and completes this part of the proof.
- To prove the “only if” part of the theorem, let $\hat{y} \in \hat{Y}(p)$ be a profit maximizing choice for the representative firm.
Proof of Theorem: The Representative Firm III

• Then, since $\hat{Y}(p) \subset Y$, we have that
  \[ \hat{y} = \sum_{f \in \mathcal{F}} y^f \]
  for some $y^f \in Y^f$ for each $f \in \mathcal{F}$.

• Let $\hat{y}^f \in \hat{Y}^f(p)$. Then,
  \[ \sum_{f \in \mathcal{F}} p \cdot y^f \leq \sum_{f \in \mathcal{F}} p \cdot \hat{y}^f, \]
  which implies that
  \[ p \cdot \hat{y} \leq p \cdot \sum_{f \in \mathcal{F}} \hat{y}^f. \quad (11) \]

• Since, by hypothesis, $\sum_{f \in \mathcal{F}} \hat{y}^f \in Y$ and $\hat{y} \in \hat{Y}(p)$, we also have
  \[ p \cdot \hat{y} \geq p \cdot \sum_{f \in \mathcal{F}} \hat{y}^f. \]
Therefore, inequality (11) must hold with equality, so that
\[ p \cdot y^f = p \cdot \hat{y}^f, \]
for each \( f \in F \), and thus \( y^f \in \hat{Y}^f(p) \). This completes the proof of the theorem.
The Representative Firm II

- Why such a difference between representative household and representative firm assumptions? Income effects.
- Changes in prices create income effects, which affect different households differently.
- No income effects in producer theory, so the representative firm assumption is without loss of any generality.
- Does not mean that heterogeneity among firms is uninteresting or unimportant.
- Many models of endogenous technology feature productivity differences across firms, and firms’ attempts to increase their productivity relative to others will often be an engine of economic growth.
Problem Formulation I

- Discrete time infinite-horizon economy and suppose that the economy admits a representative household.
- Once again ignoring uncertainty, the representative household has the $t = 0$ objective function

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad (12)$$

with a discount factor of $\beta \in (0, 1)$.
- In continuous time, this utility function of the representative household becomes

$$\int_{0}^{\infty} \exp(-\rho t) u(c(t)) \, dt \quad (13)$$

where $\rho > 0$ is now the discount rate of the individuals.
Problem Formulation II

- Where does the exponential form of the discounting in (13) come from?
- Calculate the value of $1 in $T$ periods, and divide the interval $[0, T]$ into $T/\Delta t$ equally-sized subintervals.
- Let the interest rate in each subinterval be equal to $\Delta t \cdot r$.
- Key: $r$ is multiplied by $\Delta t$, otherwise as we vary $\Delta t$, we would be changing the interest rate.
- Using the standard compound interest rate formula, the value of $1 in $T$ periods at this interest rate is
  \[ v(T \mid \Delta t) \equiv (1 + \Delta t \cdot r)^{T/\Delta t}. \]
- Now we want to take the continuous time limit by letting $\Delta t \to 0$,
  \[ v(T) \equiv \lim_{\Delta t \to 0} v(T \mid \Delta t) \equiv \lim_{\Delta t \to 0} (1 + \Delta t \cdot r)^{T/\Delta t}. \]
Thus
\[
\nu(T) \equiv \exp \left[ \lim_{\Delta t \to 0} \ln \left( 1 + \Delta t \cdot r \right)^{T/\Delta t} \right]
\]
\[
= \exp \left[ \lim_{\Delta t \to 0} \frac{T}{\Delta t} \ln \left( 1 + \Delta t \cdot r \right) \right].
\]

The term in square brackets has a limit on the form 0/0.
Write this as and use L’Hospital’s rule:
\[
\lim_{\Delta t \to 0} \frac{\ln \left( 1 + \Delta t \cdot r \right)}{\Delta t/T} = \lim_{\Delta t \to 0} \frac{r/(1 + \Delta t \cdot r)}{1/T} = rT,
\]

Therefore,
\[
\nu(T) = \exp (rT).
\]

Conversely, $1 in \ T$ periods from now, is worth \( \exp (-rT) \) today.
Same reasoning applies to utility: utility from \( c(t) \) in \( t \) evaluated at time 0 is \( \exp (-\rho t) u(c(t)) \), where \( \rho \) is (subjective) discount rate.
Welfare Theorems I

- There should be a close connection between Pareto optima and competitive equilibria.
- Start with models that have a finite number of consumers, so $H$ is finite.
- However, allow an infinite number of commodities.
- Results here have analogs for economies with a continuum of commodities, but focus on countable number of commodities.
- Let commodities be indexed by $j \in \mathbb{N}$ and $x^i \equiv \left\{ x^i_j \right\}_{j=0}^{\infty}$ be the consumption bundle of household $i$, and $\omega^i \equiv \left\{ \omega^i_j \right\}_{j=0}^{\infty}$ be its endowment bundle.
- Assume feasible $x^i$'s must belong to some consumption set $X^i \subset \mathbb{R}_+^\infty$.
- Most relevant interpretation for us is that at each date $j = 0, 1, \ldots$, each individual consumes a finite dimensional vector of products.
Welfare Theorems II

- Thus \( x_i \in X_i \subset \mathbb{R}_+^K \) for some integer \( K \).
- Consumption set introduced to allow cases where individual may not have negative consumption of certain commodities.
- Let \( X \equiv \prod_{i \in H} X_i \) be the Cartesian product of these consumption sets, the aggregate consumption set of the economy.
- Also use the notation \( x \equiv \{x^i\}_{i \in H} \) and \( \omega \equiv \{\omega^i\}_{i \in H} \) to describe the entire consumption allocation and endowments in the economy.
- Feasibility requires that \( x \in X \).
- Each household in \( H \) has a well defined preference ordering over consumption bundles.
- This preference ordering can be represented by a relationship \( \preceq_i \) for household \( i \), such that \( x' \preceq_i x \) implies that household \( i \) weakly prefers \( x' \) to \( x \).
Suppose that preferences can be represented by $u^i : X^i \to \mathbb{R}$, such that whenever $x' \succeq_i x$, we have $u^i (x') \geq u^i (x)$.

The domain of this function is $X^i \subset \mathbb{R}^\infty_+$. Let $u \equiv \{u^i\}_{i \in H}$ be the set of utility functions.

Production side: finite number of firms represented by $\mathcal{F}$

Each firm $f \in \mathcal{F}$ is characterized by production set $Y^f$, specifies levels of output firm $f$ can produce from specified levels of inputs.

I.e., $y^f \equiv \left\{y^f_j\right\}_{j=0}^\infty$ is a feasible production plan for firm $f$ if $y^f \in Y^f$.

E.g., if there were only labor and a final good, $Y^f$ would include pairs $(-l, y)$ such that with labor input $l$ the firm can produce at most $y$. 

Economic Growth

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Welfare Theorems IV

- Take each $Y^f$ to be a cone, so that if $y \in Y^f$, then $\lambda y \in Y^f$ for any $\lambda \in \mathbb{R}_+$. This implies:
  1. $0 \in Y^f$ for each $f \in \mathcal{F}$;
  2. each $Y^f$ exhibits constant returns to scale.

- If there are diminishing returns to scale from some scarce factors, this is added as an additional factor of production and $Y^f$ is still a cone.

- Let $Y \equiv \prod_{f \in \mathcal{F}} Y^f$ represent the aggregate production set and $y \equiv \{y^f\}_{f \in \mathcal{F}}$ such that $y^f \in Y^f$ for all $f$, or equivalently, $y \in Y$.

- Ownership structure of firms: if firms make profits, they should be distributed to some agents

- Assume there exists a sequence of numbers (profit shares) $\theta \equiv \{\theta^i_f\}_{f \in \mathcal{F}, i \in \mathcal{H}}$ such that $\theta^i_f \geq 0$ for all $f$ and $i$, and $\sum_{i \in \mathcal{H}} \theta^i_f = 1$ for all $f \in \mathcal{F}$.

- $\theta^i_f$ is the share of profits of firm $f$ that will accrue to household $i$. 

"Omer Ozak"
Welfare Theorems V

- An economy $\mathcal{E}$ is described by $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, u, \omega, Y, X, \theta)$.
- An allocation is $(x, y)$ such that $x$ and $y$ are feasible, that is, $x \in X$, $y \in Y$, and $\sum_{i \in \mathcal{H}} x^i \leq \sum_{i \in \mathcal{H}} \omega^i + \sum_{f \in \mathcal{F}} y^f$ for all $j \in \mathbb{N}$.
- A price system is a sequence $p \equiv \{p_j\}_{j=0}^{\infty}$, such that $p_j \geq 0$ for all $j$.
- We can choose one of these prices as the numeraire and normalize it to 1.
- Also define $p \cdot x$ as the inner product of $p$ and $x$, i.e., $p \cdot x \equiv \sum_{j=0}^{\infty} p_j x_j$. 
Welfare Theorems VI

Definition A competitive equilibrium for the economy \( E \equiv (H, F, u, \omega, Y, X, \theta) \) is given by an allocation \( (x^* = \{x^*_i\}_{i \in H}, y^* = \{y^*_f\}_{f \in F}) \) and a price system \( p^* \) such that

1. The allocation \((x^*, y^*)\) is feasible, i.e., \( x^*_i \in X^i \) for all \( i \in H \), \( y^*_f \in Y^f \) for all \( f \in F \) and
\[
\sum_{i \in H} x^*_i \leq \sum_{i \in H} \omega^i + \sum_{f \in F} y^*_f \text{ for all } j \in \mathbb{N}.
\]

2. For every firm \( f \in F \), \( y^*_f \) maximizes profits, i.e.,
\[
p^* \cdot y^*_f \geq p^* \cdot y \text{ for all } y \in Y^f.
\]

3. For every consumer \( i \in H \), \( x^*_i \) maximizes utility, i.e.,
\[
u^i (x^*_i) \geq u^i (x) \text{ for all } x \text{ s.t. } x \in X^i \text{ and } p^* \cdot x \leq p^* \cdot x^*_i.
\]
Establish existence of competitive equilibrium with finite number of commodities and standard convexity assumptions is straightforward.

With infinite number of commodities, somewhat more difficult and requires more sophisticated arguments.

**Definition**  A feasible allocation \((x, y)\) for economy 
\[ \mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, u, \omega, Y, X, \theta) \] is *Pareto optimal* if there exists no other feasible allocation \((\hat{x}, \hat{y})\) such that \(\hat{x}^i \in X^i\) for all \(i \in \mathcal{H}\), \(\hat{y}^f \in Y^f\) for all \(f \in \mathcal{F}\),

\[
\sum_{i \in \mathcal{H}} \hat{x}^i_j \leq \sum_{i \in \mathcal{H}} \omega^i_j + \sum_{f \in \mathcal{F}} \hat{y}^f_j \quad \text{for all } j \in \mathbb{N},
\]

and

\[
u^i(\hat{x}^i) \geq u^i(x^i) \quad \text{for all } i \in \mathcal{H}
\]

with at least one strict inequality.
Welfare Theorems VIII

Definition  Household $i \in \mathcal{H}$ is locally non-satiated if at each $x^i$, $u^i(x^i)$ is strictly increasing in at least one of its arguments at $x^i$ and $u^i(x^i) < \infty$.

Latter requirement already implied by the fact that $u^i : X^i \to \mathbb{R}$.

Theorem  (First Welfare Theorem I) Suppose that $(x^*, y^*, p^*)$ is a competitive equilibrium of economy $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, u, \omega, Y, X, \theta)$ with $\mathcal{H}$ finite. Assume that all households are locally non-satiated. Then $(x^*, y^*)$ is Pareto optimal.
To obtain a contradiction, suppose that there exists a feasible \((\hat{x}, \hat{y})\) such that \(u^i(\hat{x}^i) \geq u^i(x^i)\) for all \(i \in \mathcal{H}\) and \(u^i(\hat{x}^i) > u^i(x^i)\) for all \(i \in \mathcal{H}'\), where \(\mathcal{H}'\) is a non-empty subset of \(\mathcal{H}\).

Since \((x^*, y^*, p^*)\) is a competitive equilibrium, it must be the case that for all \(i \in \mathcal{H}\),

\[
p^* \cdot \hat{x}^i \geq p^* \cdot x^{i*}
\]

\[
= p^* \cdot \left(\omega^i + \sum_{f \in \mathcal{F}} \theta^i f y^f\right)
\]

and for all \(i \in \mathcal{H}'\),

\[
p^* \cdot \hat{x}^i > p^* \cdot \left(\omega^i + \sum_{f \in \mathcal{F}} \theta^i f y^f\right).
\]
Proof of First Welfare Theorem II

- Second inequality follows immediately in view of the fact that $x^{i\ast}$ is the utility maximizing choice for household $i$, thus if $\hat{x}^i$ is strictly preferred, then it cannot be in the budget set.

- First inequality follows with a similar reasoning. Suppose that it did not hold.

- Then by the hypothesis of local-satiation, $u^i$ must be strictly increasing in at least one of its arguments, let us say the $j'$th component of $x$.

- Then construct $\hat{x}^i(\varepsilon)$ such that $\hat{x}^i_j(\varepsilon) = \hat{x}^i_j$ and $\hat{x}^i_{j'}(\varepsilon) = \hat{x}^i_{j'} + \varepsilon$.

- For $\varepsilon \downarrow 0$, $\hat{x}^i(\varepsilon)$ is in household $i$’s budget set and yields strictly greater utility than the original consumption bundle $x^i$, contradicting the hypothesis that household $i$ was maximizing utility.

- Note local non-satiation implies that $u^i(x^i) < \infty$, and thus the right-hand sides of (14) and (15) are finite.
Proof of First Welfare Theorem III

- Now summing over (14) and (15), we have

\[ p^* \cdot \sum_{i \in H} \hat{x}_i > p^* \cdot \sum_{i \in H} \left( \omega^i + \sum_{f \in F} \theta^i_f y^f \right), \tag{16} \]

\[ = p^* \cdot \left( \sum_{i \in H} \omega^i + \sum_{f \in F} y^f \right), \]

- Second line uses the fact that the summations are finite, can change the order of summation, and that by definition of shares \( \sum_{i \in H} \theta^i_f = 1 \) for all \( f \).

- Finally, since \( y^* \) is profit-maximizing at prices \( p^* \), we have that

\[ p^* \cdot \sum_{f \in F} y^{f*} \geq p^* \cdot \sum_{f \in F} y^f \text{ for any } \{y^f\}_{f \in F} \text{ with } y^f \in Y^f \text{ for all } f \in F. \tag{17} \]
Proof of First Welfare Theorem IV

- However, by feasibility of $\hat{x}_i$ (Definition above, part 1), we have

$$\sum_{i \in \mathcal{H}} \hat{x}_j^i \leq \sum_{i \in \mathcal{H}} \omega_j^i + \sum_{f \in \mathcal{F}} \hat{y}_j^f,$$

- Therefore, by multiplying both sides by $p^*$ and exploiting (17),

$$p^* \cdot \sum_{i \in \mathcal{H}} \hat{x}_j^i \leq p^* \cdot \left( \sum_{i \in \mathcal{H}} \omega_j^i + \sum_{f \in \mathcal{F}} \hat{y}_j^f \right) \leq p^* \cdot \left( \sum_{i \in \mathcal{H}} \omega_j^i + \sum_{f \in \mathcal{F}} y_j^f \right),$$

- Contradicts (16), establishing that any competitive equilibrium allocation $(x^*, y^*)$ is Pareto optimal.
Proof of the First Welfare Theorem based on two intuitive ideas.

1. If another allocation Pareto dominates the competitive equilibrium, then it must be non-affordable in the competitive equilibrium.
2. Profit-maximization implies that any competitive equilibrium already contains the maximal set of affordable allocations.

Note it makes no convexity assumption.

Also highlights the importance of the feature that the relevant sums exist and are finite.

Otherwise, the last step would lead to the conclusion that “∞ < ∞”.

That these sums exist followed from two assumptions: finiteness of the number of individuals and non-satiation.
Welfare Theorems X

Theorem (First Welfare Theorem II) Suppose that \((x^*, y^*, p^*)\) is a competitive equilibrium of the economy 
\(\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, u, \omega, Y, X, \theta)\) with \(\mathcal{H}\) countably infinite. Assume that all households are locally non-satiated and that 
\[ p^* \cdot \omega^* = \sum_{i \in \mathcal{H}} \sum_{j=0}^{\infty} p_j^* \omega_j^i < \infty. \]
Then \((x^*, y^*, p^*)\) is Pareto optimal.

Proof:

- Same as before but now local non-satiation does not guarantee summations are finite (16), since we sum over an infinite number of households.
- But since endowments are finite, the assumption that 
\[ \sum_{i \in \mathcal{H}} \sum_{j=0}^{\infty} p_j^* \omega_j^i < \infty \]
ensures that the sums in (16) are indeed finite.
Welfare Theorems X

- Second Welfare Theorem (converse to First): whether or not $\mathcal{H}$ is finite is not as important as for the First Welfare Theorem.
- But requires assumptions such as the convexity of consumption and production sets and preferences, and additional requirements because it contains an “existence of equilibrium argument”.
- Recall that the consumption set of each individual $i \in \mathcal{H}$ is $X^i \subset \mathbb{R}^\infty_+$. A typical element of $X^i$ is $x^i = (x^i_1, x^i_2, ...)$, where $x^i_t$ can be interpreted as the vector of consumption of individual $i$ at time $t$.
- Similarly, a typical element of the production set of firm $f \in \mathcal{F}$, $Y^f$, is $y^f = (y^f_1, y^f_2, ...)$. Let us define $x^i[T] = (x^i_0, x^i_1, x^i_2, ..., x^i_T, 0, 0, ...)$ and $y^f[T] = (y^f_0, y^f_1, y^f_2, ..., y^f_T, 0, 0, ...)$. It can be verified that $\lim_{T \to \infty} x^i[T] = x^i$ and $\lim_{T \to \infty} y^f[T] = y^f$ in the product topology.
Second Welfare Theorem I

Theorem

Consider a Pareto optimal allocation \((x^{**}, y^{**})\) in an economy described by \(\omega, \{Y^f\}_{f \in F}, \{X^i\}_{i \in \mathcal{H}}, \text{ and } \{u^i(\cdot)\}_{i \in \mathcal{H}}\). Suppose all production and consumption sets are convex, all production sets are cones, and all \(\{u^i(\cdot)\}_{i \in \mathcal{H}}\) are continuous and quasi-concave and satisfy local non-satiation. Suppose also that \(0 \in X^i\), that for each \(x, x' \in X^i\) with \(u^i(x) > u^i(x')\) for all \(i \in \mathcal{H}\), there exists \(\bar{T}\) such that \(u^i(x[T]) > u^i(x')\) for all \(T \geq \bar{T}\) and for all \(i \in \mathcal{H}\), and that for each \(y \in Y^f\), there exists \(\tilde{T}\) such that \(y[T] \in Y^f\) for all \(T \geq \tilde{T}\) and for all \(f \in F\). Then this allocation can be decentralized as a competitive equilibrium.
Second Welfare Theorem II

Theorem

(continued) In particular, there exist $p^{**}$ and $(\omega^{**}, \theta^{**})$ such that

1. $\omega^{**}$ satisfies $\omega = \sum_{i \in H} \omega^{i**}$;
2. for all $f \in F$,
   $$p^{**} \cdot y^{f**} \geq p^{**} \cdot y \text{ for all } y \in Y^{f};$$
3. for all $i \in H$,
   $$\text{if } x^{i} \in X^{i} \text{ involves } u^{i}(x^{i}) > u^{i}(x^{i**}), \text{ then } p^{**} \cdot x^{i} \geq p^{**} \cdot w^{i**},$$
   where $w^{i**} \equiv \omega^{i**} + \sum_{f \in F} \theta^{i**}_{f} y^{f**}$.

Moreover, if $p^{**} \cdot w^{**} > 0$ [i.e., $p^{**} \cdot w^{i**} > 0$ for each $i \in H$], then economy $E$ has a competitive equilibrium $(x^{**}, y^{**}, p^{**})$. 
Welfare Theorems XII

- Notice:
  - if instead we had a finite commodity space, say with \( K \) commodities, then the hypothesis that \( 0 \in X^i \) for each \( i \in \mathcal{H} \) and \( x, x' \in X^i \) with \( u^i(x) > u^i(x') \), there exists \( \bar{T} \) such that \( u^i(x[T]) > u^i(x'[T]) \) for all \( T \geq \bar{T} \) and all \( i \in \mathcal{H} \) (and also that there exists \( \tilde{T} \) such that if \( y \in Y^f \), then \( y[T] \in Y^f \) for all \( T \geq \tilde{T} \) and all \( f \in \mathcal{F} \) would be satisfied automatically, by taking \( \bar{T} = \tilde{T} = K \).
  - Condition not imposed in Second Welfare Theorem in economies with a finite number of commodities.
  - In dynamic economies, its role is to ensure that changes in allocations at very far in the future should not have a large effect.

- The conditions for the Second Welfare Theorem are more difficult to satisfy than those for the First.
- Also the more important of the two theorems: stronger results that any Pareto optimal allocation can be decentralized.
Welfare Theorems XIII

- Immediate corollary is an existence result: a competitive equilibrium must exist.
- Motivates many to look for the set of Pareto optimal allocations instead of explicitly characterizing competitive equilibria.
- Real power of the Theorem in dynamic macro models comes when we combine it with models that admit a representative household.
- Enables us to characterize *the optimal growth allocation* that maximizes the utility of the representative household and assert that this will correspond to a competitive equilibrium.
Sketch of the Proof of SWT I

- First, I establish that there exists a price vector $p^{**}$ and an endowment and share allocation $(\omega^{**}, \theta^{**})$ that satisfy conditions 1-3.
- This has two parts.
- (Part 1) This part follows from the Geometric Hahn-Banach Theorem.
- Define the “more preferred” sets for each $i \in \mathcal{H}$:

$$P^i = \{ x^i \in X^i : u^i(x^i) > u^i(x^{i**}) \}.$$ 

- Clearly, each $P^i$ is convex.
- Let $P = \sum_{i \in \mathcal{H}} P^i$ and $Y' = \sum_{f \in \mathcal{F}} Y^f + \{ \omega \}$, where recall that $\omega = \sum_{i \in \mathcal{H}} \omega^{i**}$, so that $Y'$ is the sum of the production sets shifted by the endowment vector.
- Both $P$ and $Y'$ are convex (since each $P^i$ and each $Y^f$ are convex).
Consider the sequences of production plans for each firm to be subsets of $\ell^K_\infty$, i.e., vectors of the form $y^f = (y^f_0, y^f_1, \ldots)$, with each $y^f_j \in \mathbb{R}^K_+$. Moreover, since each production set is a cone, $Y' = \sum_{f \in F} Y^f + \{\omega\}$ has an interior point.

Moreover, let $x^{**} = \sum_{i \in H} x^{i**}$. By feasibility and local non-satiation, $x^{**} = \sum_{f \in F} y^{i**} + \omega$. Then $x^{**} \in Y'$ and also $x^{**} \in \overline{P}$ (where $\overline{P}$ is the closure of $P$).

Next, observe that $P \cap Y' = \emptyset$. Otherwise, there would exist $\tilde{y} \in Y'$, which is also in $P$.

This implies that if distributed appropriately across the households, $\tilde{y}$ would make all households equally well off and at least one of them would be strictly better off.
Sketch of the Proof of SWT III

- I.e., by the definition of the set \( P \), there would exist \( \{ \tilde{x}^i \}_{i \in \mathcal{H}} \) such that \( \sum_{i \in \mathcal{H}} \tilde{x}^i = \tilde{y} \), \( \tilde{x}^i \in X^i \), and \( u^i (\tilde{x}^i) \geq u^i (x^{i**}) \) for all \( i \in \mathcal{H} \) with at least one strict inequality.

- This would contradict the hypothesis that \((x^{**}, y^{**})\) is a Pareto optimum.

- Since \( Y' \) has an interior point, \( P \) and \( Y' \) are convex, and \( P \cap Y' = \emptyset \), Geometric Theorem implies that there exists a nonzero continuous linear functional \( \phi \) such that

\[
\phi (y) \leq \phi (x^{**}) \leq \phi (x) \quad \text{for all } y \in Y' \text{ and all } x \in P. \tag{18}
\]

- (Part 2) We next need to show that this linear functional can be interpreted as a price vector (i.e., that it does have an inner product representation).

- Let, \( \bar{\phi} (x) = \lim_{T \to \infty} \phi (x [T]) \).
Sketch of the Proof of SWT IV

- Then, first note that if $\phi(x)$ is a continuous linear functional, then $\bar{\phi}(x) = \sum_{j=0}^{\infty} \bar{\phi}_j(x_j)$ is also a linear functional, where each $\bar{\phi}_j(x_j)$ is a linear functional on $X_j \subset \mathbb{R}_+^K$.

- Second claim follows from the fact that $\phi(x[T])$ is bounded above by $\|\phi\| \cdot \|x\|$, where $\|\phi\|$ denotes the norm of the functional $\phi$ and is thus finite.

- Clearly, $\|x\|$ is also finite.

- Moreover, since each element of $x$ is nonnegative, $\{\phi(x[t])\}$ is a monotone sequence, thus $\lim_{T \to \infty} \phi(x[T])$ converges and we denote the limit by $\bar{\phi}(x)$.

- Moreover, this limit is a bounded functional and therefore from Continuity of Linear Function Theorem, it is continuous.
Sketch of the Proof of SWT V

The first claim follows from the fact that since \( x_j \in X_j \subset \mathbb{R}_+^K \), we can define a continuous linear functional on the dual of \( X_j \) by
\[
\bar{\phi}_j (x_j) = \phi (\tilde{x}^j) = \sum_{s=1}^{K} p_{j,s}^* x_{j,s},
\]
where \( \tilde{x}^j = (0, 0, ..., x_j, 0, ...) \) [i.e., \( \tilde{x}^j \) has \( x_j \) as \( j \)th element and zeros everywhere else].

Then clearly,
\[
\bar{\phi} (x) = \sum_{j=0}^{\infty} \bar{\phi}_j (x_j) = \sum_{s=0}^{\infty} p_{s}^{**} x_s = p^{**} \cdot x.
\]

To complete this part of the proof, we only need to show that
\[
\bar{\phi} (x) = \sum_{j=0}^{\infty} \bar{\phi}_j (x_j)
\]
can be used instead of \( \phi \) as the continuous linear functional in (18).
Sketch of the Proof of SWT VI

- This follows immediately from the hypothesis that $0 \in X^i$ for each $i \in \mathcal{H}$ and that there exists $\bar{T}$ such that for any $x, x' \in X^i$ with $u^i(x) > u^i(x')$, $u^i(x[T]) > u^i(x'[T])$ for all $T \geq \bar{T}$ and for all $i \in \mathcal{H}$, and that there exists $\tilde{T}$ such that if $y \in Y^f$, then $y[T] \in Y^f$ for all $T \geq \tilde{T}$ and for all $f \in \mathcal{F}$.

- In particular, take $T' = \max \{ \bar{T}, \tilde{T} \}$ and fix $x \in P$.

- Since $x$ has the property that $u^i(x^i) > u^i(x^{i**})$ for all $i \in \mathcal{H}$, we also have that $u^i(x^i[T]) > u^i(x^{i**}[T])$ for all $i \in \mathcal{H}$ and $T \geq T'$.

- Therefore,

$$\phi(x^{**}[T]) \leq \phi(x[T]) \quad \text{for all } x \in P.$$

- Now taking limits,

$$\bar{\phi}(x^{**}) \leq \bar{\phi}(x) \quad \text{for all } x \in P.$$
Sketch of the Proof of SWT VII

A similar argument establishes that $\phi(x^{**}) \geq \phi(y)$ for all $y \in Y'$, so that $\phi(x)$ can be used as the continuous linear functional separating $P$ and $Y'$.

Since $\phi_j(x_j)$ is a linear functional on $X_j \subset \mathbb{R}_+^K$, it has an inner product representation, $\phi_j(x_j) = p_{j}^{**} \cdot x_j$ and therefore so does $\phi(x) = \sum_{j=0}^{\infty} \phi_j(x_j) = p^{**} \cdot x$.

Parts 1 and 2 have therefore established that there exists a price vector (functional) $p^{**}$ such that conditions 2 and 3 hold.

Condition 1 is satisfied by construction.

Condition 2 is sufficient to establish that all firms maximize profits at the price vector $p^{**}$.

To show that all consumers maximize utility at the price vector $p^{**}$, use the hypothesis that $p^{**} \cdot w^{i**} > 0$ for each $i \in H$. 
Welfare Theorems
Sketch of the Proof

Sketch of the Proof of SWT VIII

- We know from Condition 3 that if \( x^i \in X^i \) involves \( u^i (x^i) > u^i (x^{i**}) \), then \( p^{**} \cdot x^i \geq p^{**} \cdot w^{i**} \).
- This implies that if there exists \( x^i \) that is strictly preferred to \( x^{i**} \) and satisfies \( p^{**} \cdot x^i = p^{**} \cdot w^{i**} \) (which would amount to the consumer not maximizing utility), then there exists \( x^i - \varepsilon \) for \( \varepsilon \) small enough, such that \( u^i (x^i - \varepsilon) > u^i (x^{i**}) \), then \( p^{**} \cdot (x^i - \varepsilon) < p^{**} \cdot w^{i**} \), thus violating Condition 3.
- Therefore, consumers also maximize utility at the price \( p^{**} \), establishing that \((x^{**}, y^{**}, p^{**})\) is a competitive equilibrium. \( \square \)
Sequential Trading I

- Standard general equilibrium models assume all commodities are traded at a given point in time—and once and for all.
- When trading same good in different time periods or states of nature, trading once and for all less reasonable.
- In models of economic growth, typically assume trading takes place at different points in time.
- But with complete markets, sequential trading gives the same result as trading at a single point in time.
- *Arrow-Debreu equilibrium* of dynamic general equilibrium model: all households trading at $t = 0$ and purchasing and selling irrevocable claims to commodities indexed by date and state of nature.
- Sequential trading: separate markets at each $t$, households trading labor, capital and consumption goods in each such market.
- With complete markets (and time consistent preferences), both are equivalent.
Sequential Trading II

- **(Basic) Arrow Securities**: means of transferring resources across different dates and different states of nature.
- Households can trade Arrow securities and then use these securities to purchase goods at different dates or after different states of nature.
- Reason why both are equivalent:
  - by definition of competitive equilibrium, households correctly anticipate all the prices and purchase sufficient Arrow securities to cover the expenses that they will incur.
- Instead of buying claims at time $t = 0$ for $x_{i,t'}^h$ units of commodity $i = 1, ..., N$ at date $t'$ at prices $(p_{1,t'}, ..., p_{N,t'})$, sufficient for household $h$ to have an income of $\sum_{i=1}^N p_{i,t'}x_{i,t'}^h$ and know that it can purchase as many units of each commodity as it wishes at time $t'$ at the price vector $(p_{1,t'}, ..., p_{N,t'})$.
- Consider a dynamic exchange economy running across periods $t = 0, 1, ..., T$, possibly with $T = \infty$. 
Sequential Trading III

- There are $N$ goods at each date, denoted by $(x_{1,t}, \ldots, x_{N,t})$.
- Let the consumption of good $i$ by household $h$ at time $t$ be denoted by $x_{i,t}^h$.
- Goods are perishable, so that they are indeed consumed at time $t$.
- Each household $h \in \mathcal{H}$ has a vector of endowment $(\omega_{1,t}^h, \ldots, \omega_{N,t}^h)$ at time $t$, and preferences

$$\sum_{t=0}^{T} \beta_{h}^{t} u^{h} \left( x_{1,t}^h, \ldots, x_{N,t}^h \right),$$

for some $\beta_{h} \in (0, 1)$.
- These preferences imply no externalities and are time consistent.
- All markets are open and competitive.
- Let an Arrow-Debreu equilibrium be given by $(p^*, x^*)$, where $x^*$ is the complete list of consumption vectors of each household $h \in \mathcal{H}$. 
That is,

\[ x^* = (x_{1,0}, \ldots, x_{N,0}, \ldots, x_{1,T}, \ldots, x_{N,T}) , \]

with \( x_{i,t} = \{x^h_{i,t}\}_{h \in \mathcal{H}} \) for each \( i \) and \( t \).

- \( p^* \) is the vector of complete prices

\[ p^* = (p^*_{1,0}, \ldots, p^*_{N,0}, \ldots, p^*_{1,T}, \ldots, p^*_{N,T}) , \]

with \( p^*_{1,0} = 1 \).

- Arrow-Debreu equilibrium: trading only at \( t = 0 \) and choose allocation that satisfies

\[ \sum_{t=0}^{T} \sum_{i=1}^{N} p^*_{i,t} x^h_{i,t} \leq \sum_{t=0}^{T} \sum_{i=1}^{N} p^*_{i,t} \omega^h_{i,t} \text{ for each } h \in \mathcal{H}. \]

- Market clearing then requires

\[ \sum_{h \in \mathcal{H}} x^h_{i,t} \leq \sum_{h \in \mathcal{H}} \omega^h_{i,t} \text{ for each } i = 1, \ldots, N \text{ and } t = 0, 1, \ldots, T. \]
Sequential Trading V

- Equilibrium with sequential trading:
  - Markets for goods dated $t$ open at time $t$.
  - There are $T$ bonds—Arrow securities—in zero net supply that can be traded at $t = 0$.
  - Bond indexed by $t$ pays one unit of one of the goods, say good $i = 1$ at time $t$.

- Prices of bonds denoted by $(q_1, \ldots, q_T)$, expressed in units of good $i = 1$ (at time $t = 0$).

- Thus a household can purchase a unit of bond $t$ at time 0 by paying $q_t$ units of good 1 and will receive one unit of good 1 at time $t$.

- Denote purchase of bond $t$ by household $h$ by $b^h_t \in \mathbb{R}$.

- Since each bond is in zero net supply, market clearing requires

\[
\sum_{h \in \mathcal{H}} b^h_t = 0 \text{ for each } t = 0, 1, \ldots, T.
\]
Sequential Trading VI

- Each individual uses his endowment plus (or minus) the proceeds from the corresponding bonds at each date $t$.
- Convenient (and possible) to choose a separate numeraire for each date $t$, $p_{1,t}^{**} = 1$ for all $t$.
- Therefore, the budget constraint of household $h \in \mathcal{H}$ at time $t$, given equilibrium $(p^{**}, q^{**})$:

$$
\sum_{i=1}^{N} p_{i,t}^{**} x_{i,t}^h \leq \sum_{i=1}^{N} p_{i,t}^{**} \omega_{i,t}^h + q_t^{**} b_t^h \text{ for } t = 0, 1, \ldots, T,
$$

(19)

together with the constraint

$$
\sum_{t=0}^{T} q_t^{**} b_t^h \leq 0
$$

with the normalization that $q_0^{**} = 1$. 
Sequential Trading VII

Let equilibrium with sequential trading be \((p^{**}, q^{**}, x^{**}, b^{**})\).

**Theorem** *(Sequential Trading)* For the above-described economy, if \((p^{*}, x^{*})\) is an Arrow-Debreu equilibrium, then there exists a sequential trading equilibrium \((p^{**}, q^{**}, x^{**}, b^{**})\), such that \(x^{*} = x^{**}\), \(p^{**}_{i,t} = p^{*}_{i,t}/p^{1}_{1,t}\) for all \(i\) and \(t\) and \(q^{**}_{t} = p^{1}_{1,t}\) for all \(t > 0\). Conversely, if \((p^{**}, q^{**}, x^{**}, b^{**})\) is a sequential trading equilibrium, then there exists an Arrow-Debreu equilibrium \((p^{*}, x^{*})\) with \(x^{*} = x^{**}\), \(p^{*}_{i,t} = p^{**}_{i,t}p^{1}_{1,t}\) for all \(i\) and \(t\), and \(p^{1}_{1,t} = q^{**}_{t}\) for all \(t > 0\).

Focus on economies with sequential trading and assume that there exist Arrow securities to transfer resources across dates.

These securities might be riskless bonds in zero net supply, or without uncertainty, role typically played by the capital stock.

Also typically normalize the price of one good at each date to 1.

Hence interest rates are key relative prices in dynamic models.
Optimal Growth in Discrete Time I

- Economy characterized by an aggregate production function, and a representative household.
- Optimal growth problem in discrete time with no uncertainty, no population growth and no technological progress:

\[
\max_{\{c_t, k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)
\]  

subject to

\[
k_{t+1} = f(k_t) + (1 - \delta) k_t - c_t,
\]  

\[k_t \geq 0 \text{ and given } k_0 > 0.\]

- Initial level of capital stock is \(k_0\), but this gives a single initial condition.
Solution will correspond to two difference equations, thus need another boundary condition.

Will come from the optimality of a dynamic plan in the form of a transversality condition.

Can be solved in a number of different ways: e.g., infinite dimensional Lagrangian, but the most convenient is by dynamic programming.

Note even if we wished to bypass the Second Welfare Theorem and directly solve for competitive equilibria, we would have to solve a problem similar to the maximization of (20) subject to (21).
Optimal Growth in Discrete Time III

- Assuming that the representative household has one unit of labor supplied inelastically, this problem can be written as:

$$\max_{\{c_t, k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to some given $a_0$ and

$$a_{t+1} = R_t [a_t - c(t) + w_t], \quad (22)$$

- Need an additional condition so that this flow budget constraint eventually converges (i.e., so that $a_t$ should not go to negative infinity).

- Can impose a lifetime budget constraint, or augment flow budget constraint with another condition to rule out wealth going to negative infinity.
The formulation of the optimal growth problem in continuous time is very similar:

\[
\max_{[c(t),k(t)]_{t=0}^{\infty}} \int_{0}^{\infty} \exp(-\rho t) u(c(t)) \, dt
\]  

subject to

\[
\dot{k}(t) = f(k(t)) - c(t) - \delta k(t),
\]  

\(k(t) \geq 0\) and given \(k(0) = k_0 > 0\).

The objective function (23) is the direct continuous-time analog of (20), and (24) gives the resource constraint of the economy, similar to (21) in discrete time.

Again, lacks one boundary condition which will come from the transversality condition.

Most convenient way of characterizing the solution to this problem is via optimal control theory.
Models we study in this book are examples of more general dynamic general equilibrium models.

First and the Second Welfare Theorems are essential.

The most general class of dynamic general equilibrium models are not tractable enough to derive sharp results about economic growth.

Need simplifying assumptions, the most important one being the representative household assumption.