Optimal Control

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Macroeconomics II

Section 1

Review of the Theory of Optimal Control

Review of the Theory of Optimal Control I

- We have seen how to solve a countably infinite-dimensional optimization problem using Dynamic Programming and Bellman's Operator both analytically and computationally.
- Now let us review of basic results in dynamic optimization in continuous time—particularly the *optimal control* approach.
- New mathematical issues: even with a finite horizon, the maximization is with respect to an infinite-dimensional object (an entire function, $y : [t_0, t_1] \to \mathbb{R}$).
- Requires brief review of *calculus of variations* and from the theory of optimal control.

Review of the Theory of Optimal Control II

- Canonical problem

$$\max_{\mathbf{x}(t),\mathbf{y}(t)} W\left(\mathbf{x}\left(t\right),\mathbf{y}\left(t\right)\right) \equiv \int_0^{t_1} f\left(t,\mathbf{x}\left(t\right),\mathbf{y}\left(t\right)\right) dt$$

subject to

$$\dot{\mathbf{x}}\left(t\right) = \mathbf{G}\left(t,\mathbf{x}\left(t\right),\mathbf{y}\left(t\right)\right)$$

and

$$\mathbf{x}(t) \in \mathcal{X}(t),\ \mathbf{y}\left(t\right) \in \mathcal{Y}\left(t\right) \text{ for all } t \geq 0, \mathbf{x}\left(0\right)=\mathbf{x}_0$$

Review of the Theory of Optimal Control III

- For each t, $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are finite-dimensional vectors (i.e., $\mathbf{x}(t) \in \mathbb{R}^{K_x}$ and $\mathbf{y}(t) \in \mathbb{R}^{K_y}$, where K_x and K_y are integers).
- Refer to $\mathbf{x}$ as the *state* variable, governed by a vector-valued differential equation given behavior of *control* variables $\mathbf{y}(t)$.
- End of planning horizon t_1 can be infinity.
- Function $W(\mathbf{x}(t), \mathbf{y}(t))$: *objective function* when controls are $\mathbf{y}(t)$ and resulting state variable is summarized by $\mathbf{x}(t)$.
- Refer to f as the *instantaneous objective function* (or the payoff function) and to $\mathbf{G}$ as the *constraint function.*

Section 2

Variational Arguments

Variational Arguments I

- Consider the following finite-horizon continuous time problem

$$\max_{x(t),y(t),x_1} W\left(x\left(t\right),y\left(t\right)\right) \equiv \int_0^{t_1} f\left(t,x\left(t\right),y\left(t\right)\right) dt \tag{1}$$

subject to

$$\dot{x}\left(t\right) = g\left(t,x\left(t\right),y\left(t\right)\right) \tag{2}$$

and

$$x(t) \in \mathcal{X}(t),\ y\left(t\right) \in \mathcal{Y}\left(t\right) \text{ for all } t,\ x\left(0\right) = x_0 \text{ and } x(t_1) = x_1. \tag{3}$$

Variational Arguments II

- $x(t) \in \mathbb{R}$ is one-dimensional and its behavior is governed by the differential equation (2).
- $y(t)$ must belong to the set $\mathcal{Y}(t) \subset \mathbb{R}$.
- $\mathcal{Y}(t)$ is nonempty and convex.
- We call $(x(t), y(t))$ an *admissible pair* if they jointly satisfy (2) and (3).

Variational Arguments III

- Suppose that $W(x(t), y(t)) < \infty$ for any admissible pair $(x(t), y(t))$ and that $t_1 < \infty$.
- Two versions of the problem:
  - Terminal value problem: additional constraint $x(t_1) = x_1$,
  - Terminal free problem: x_1 is included as an additional choice variable.
- Assume that f and g are continuously differentiable functions.

Variational Arguments IV

- Difficulty lies in two features:
  1. Choosing a function $y : [0, t_1] \to \mathcal{Y}$ rather than a vector or a finite dimensional object.
  2. Constraint is a differential equation, rather than a set of inequalities or equalities (although sometimes we also have those).
- Hard to know what type of optimal policy to look for: y may be highly discontinuous function, or hit the boundary.
- In economic problems we impose structure: Continuity and Inada conditions ensure solutions are continuous and lie in the interior.
- When y is a continuous function of time and lies in the interior of the feasible set we can use variational arguments.

Variational Arguments V

- *Variational principle*: start assuming a continuous solution (function) $\hat{y}$ that lies everywhere in the interior of the set $\mathcal{Y}$ exists.
- Assume $(\hat{x}(t), \hat{y}(t))$ is an admissible pair such that $\hat{y}(\cdot)$ is continuous over $[0, t_1]$ and $\hat{y}(t) \in \text{Int}\mathcal{Y}(t)$, and

$$W(\hat{x}(t), \hat{y}(t)) \geq W(x(t), y(t))$$

  for any other admissible pair $(x(t), y(t))$.
- Note x is given by (2), so when $y(t)$ is continuous, $\dot{x}(t)$ will also be continuous, so $x(t)$ is continuously differentiable.
- When $y(t)$ is piecewise continuous, $x(t)$ will be, correspondingly, piecewise smooth.

Variational Arguments VI

- Exploit these features to derive *necessary* conditions for an optimal path.
- Consider the following *variation*

$$y\left(t,\varepsilon\right)\equiv\hat{y}\left(t\right)+\varepsilon\eta\left(t\right),$$

- $\eta\left(t\right)$ is an arbitrary *fixed* continuous function and $\varepsilon\in\mathbb{R}$ is a scalar.
- Variation: given $\eta\left(t\right)$, by varying ε, we obtain different sequences of controls.
- Some of these may be infeasible, i.e., $y\left(t,\varepsilon\right)\notin\mathcal{Y}\left(t\right)$ for some t.
- But since $\hat{y}\left(t\right)\in\text{Int}\mathcal{Y}\left(t\right)$, and a continuous function over a compact set $\left[0,t_1\right]$ is bounded, for any fixed $\eta\left(\cdot\right)$ function, can always find $\varepsilon_\eta>0$ such that

$$y\left(t,\varepsilon\right)\equiv\hat{y}\left(t\right)+\varepsilon\eta\left(t\right)\in\text{Int}\mathcal{Y}\left(t\right)$$

for all $\varepsilon\in\left[-\varepsilon_\eta,\varepsilon_\eta\right]$.

Variational Arguments VII

- Thus $y(t,\varepsilon)$ constitutes a *feasible variation*: can use variational arguments for sufficiently small ε's.
- Can ensure no small change increase the value of the objective function, although non-infinitesimal changes might lead to a higher value.
- Fix an arbitrary $\eta(\cdot)$, and define $x(t,\varepsilon)$ as the path of the state variable corresponding to the path of control variable $y(t,\varepsilon)$.
- This implies $x(t,\varepsilon)$ is given by:

$$\dot{x}(t,\varepsilon) \equiv g(t, x(t,\varepsilon), y(t,\varepsilon)) \text{ for all } t \in [0, t_1], \text{ with } x(0,\varepsilon) = x_0. \tag{4}$$

Variational Arguments VIII

- For $\varepsilon \in \left[-\varepsilon_\eta, \varepsilon_\eta\right]$, define:

$$\begin{aligned}\mathcal{W}(\varepsilon) &\equiv W(x(t,\varepsilon), y(t,\varepsilon)) \\ &= \int_0^{t_1} f(t, x(t,\varepsilon), y(t,\varepsilon))\, dt.\end{aligned} \tag{5}$$

- By the fact that $\hat{y}(t)$ is optimal, and that for $\varepsilon \in \left[-\varepsilon_\eta, \varepsilon_\eta\right]$, $y(t,\varepsilon)$ and $x(t,\varepsilon)$ are feasible,

$$\mathcal{W}(\varepsilon) \leq \mathcal{W}(0) \equiv W(x(t,0), y(t,0)) \text{ for all } \varepsilon \in \left[-\varepsilon_\eta, \varepsilon_\eta\right].$$

Variational Arguments IX

- Next, rewrite the equation (4), so that

$$g\left(t, x\left(t, \varepsilon\right), y\left(t, \varepsilon\right)\right) - \dot{x}\left(t, \varepsilon\right) \equiv 0$$

  for all $t \in [0, t_1]$.
- This implies that for *any* function $\lambda : [0, t_1] \to \mathbb{R}$,

$$\int_0^{t_1} \lambda\left(t\right)\left[g\left(t, x\left(t, \varepsilon\right), y\left(t, \varepsilon\right)\right) - \dot{x}\left(t, \varepsilon\right)\right] dt = 0, \tag{6}$$

  since the term in square brackets is identically equal to zero.
- Take $\lambda\left(\cdot\right)$ continuously differentiable.

Variational Arguments X

- Chosen suitably, this will be the *costate* variable, with similar interpretation to Lagrange multipliers.
- Adding (6) to (5), we obtain

$$\mathcal{W}(\varepsilon) \equiv \int_0^{t_1} \left\{ \begin{array}{c} f(t, x(t,\varepsilon), y(t,\varepsilon)) \\ +\lambda(t)\left[g(t, x(t,\varepsilon), y(t,\varepsilon)) - \dot{x}(t,\varepsilon)\right] \end{array} \right\} dt. \quad (7)$$

- To evaluate (7), integrate by parts $\int_0^{t_1} \lambda(t)\,\dot{x}(t,\varepsilon)\,dt$,

$$\int_0^{t_1} \lambda(t)\,\dot{x}(t,\varepsilon)\,dt = \lambda(t_1)\,x(t_1,\varepsilon) - \lambda(0)\,x_0 - \int_0^{t_1} \dot{\lambda}(t)\,x(t,\varepsilon)\,dt.$$

- Substituting this expression back into (7):

$$\begin{aligned} \mathcal{W}(\varepsilon) \equiv & \int_0^{t_1} \left[\begin{array}{c} f(t, x(t,\varepsilon), y(t,\varepsilon)) \\ +\lambda(t)\,g(t, x(t,\varepsilon), y(t,\varepsilon)) + \dot{\lambda}(t)\,x(t,\varepsilon) \end{array} \right] dt \\ & -\lambda(t_1)\,x(t_1,\varepsilon) + \lambda(0)\,x_0. \end{aligned}$$

Variational Arguments XI

- Recall f and g are continuously differentiable, and $y(t,\varepsilon)$ is continuously differentiable in ε by construction.
- Hence $x(t,\varepsilon)$ is continuously differentiable in ε.
- Differentiating the previous expression with respect to ε (making use of Leibniz's rule),

$$
\begin{aligned}
\mathcal{W}'(\varepsilon) \equiv & \int_0^{t_1} \begin{bmatrix} f_x(t, x(t,\varepsilon), y(t,\varepsilon)) \\ +\lambda(t) g_x(t, x(t,\varepsilon), y(t,\varepsilon)) + \dot{\lambda}(t) \end{bmatrix} x_\varepsilon(t,\varepsilon)\, dt \\
& + \int_0^{t_1} \begin{bmatrix} f_y(t, x(t,\varepsilon), y(t,\varepsilon)) \\ +\lambda(t) g_y(t, x(t,\varepsilon), y(t,\varepsilon)) \end{bmatrix} \eta(t)\, dt \\
& -\lambda(t_1) x_\varepsilon(t_1,\varepsilon).
\end{aligned}
$$

Variational Arguments XII

- Next evaluate this derivative at $\varepsilon = 0$:

$$\mathcal{W}'(0) \equiv \int_0^{t_1} \begin{bmatrix} f_x(t, \hat{x}(t), \hat{y}(t)) \\ +\lambda(t) g_x(t, \hat{x}(t), \hat{y}(t)) + \dot{\lambda}(t) \end{bmatrix} x_\varepsilon(t, 0)\, dt + \int_0^{t_1} \begin{bmatrix} f_y(t, \hat{x}(t), \hat{y}(t)) \\ +\lambda(t) g_y(t, \hat{x}(t), \hat{y}(t)) \end{bmatrix} \eta(t)\, dt - \lambda(t_1) x_\varepsilon(t_1, 0).$$

- $\hat{x}(t) = x(t, \varepsilon = 0)$: path of the state variable corresponding to the optimal plan, $\hat{y}(t)$.
- If there exists some $\eta(t)$ for which $\mathcal{W}'(0) \neq 0$, $W(x(t), y(t))$ can be increased; $(\hat{x}(t), \hat{y}(t))$ could not be optimal.
- Consequently, optimality requires that

$$\mathcal{W}'(0) \equiv 0 \text{ for all } \eta(t) \text{ and } \lambda(t). \tag{8}$$

Variational Arguments XIII

- $\mathcal{W}'(0)$ applies for any continuously differentiable $\lambda(t)$ function, but not all such functions $\lambda(\cdot)$ will play the role of a *costate variable* (appropriate multipliers).
- Choose the function $\lambda(\cdot)$ as a solution to the following differential equation:

$$\dot{\lambda}(t) = -\left[f_x(t, \hat{x}(t), \hat{y}(t)) + \lambda(t) g_x(t, \hat{x}(t), \hat{y}(t))\right], \quad (9)$$

  with boundary condition $\lambda(t_1) = 0$.
- Given this choice of $\lambda(t)$, $\mathcal{W}'(0) \equiv 0$ requires

$$\int_0^{t_1} \left[\begin{array}{c} f_y(t, \hat{x}(t), \hat{y}(t)) \\ +\lambda(t) g_y(t, \hat{x}(t), \hat{y}(t)) \end{array} \right] \eta(t)\, dt = 0 \text{ for all } \eta(t).$$

  (since $\eta(t)$ is arbitrary).

Variational Arguments XIV

- Therefore, it is necessary that

$$f_y\left(t, \hat{x}\left(t\right), \hat{y}\left(t\right)\right) + \lambda\left(t\right) g_y\left(t, \hat{x}\left(t\right), \hat{y}\left(t\right)\right) = 0 \tag{10}$$

- Thus necessary conditions for *interior continuous solution* to maximizing (1) subject to (2) and (3) are such that there should exist a continuously differentiable function $\lambda\left(\cdot\right)$ that satisfies (9), (10) and $\lambda\left(t_1\right) = 0$.
- The condition that $\lambda\left(t_1\right) = 0$ is the *transversality condition* of continuous time optimization problems:
    - after the planning horizon, there is no value to having more x.

Variational Arguments XV

Theorem (Necessary Conditions)

If $(\hat{x}(t), \hat{y}(t))$ is an interior solution to the maximization problem (1) subject to (2) and (3), with f and g continuously differentiable, then there exists a continuously differentiable costate function $\lambda(\cdot)$ defined over $t \in [0, t_1]$ such that (2), (9) and (10) hold, and moreover $\lambda(t_1) = 0$.

- If $x(t_1) = x_1$ is fixed, then $\lambda(t_1)$ is unrestricted. If $x(t_1) \geq x_1$, then $\lambda(t_1).(x(t_1) - x_1) = 0$
- When terminal value of the state variable, x_1, is fixed, the maximization problem is

$$\max_{x(t), y(t)} W(x(t), y(t)) \equiv \int_0^{t_1} f(t, x(t), y(t))\, dt, \qquad (11)$$

subject to (2) and (3).

Section 3

The Maximum Principle: A First Look

Subsection 1

The Hamiltonian and the Maximum Principle

The Hamiltonian and the Maximum Principle I

- Define Hamiltonian:

$$H(t, x, y, \lambda) \equiv f(t, x(t), y(t)) + \lambda(t) g(t, x(t), y(t)). \quad (12)$$

- Since f and g are continuously differentiable, so is H.
- Can rewrite the necessary condition in terms of Hamiltonians as follows

The Hamiltonian and the Maximum Principle II

Theorem (Necessary Conditions II)

Consider the problem of maximizing (1) subject to (2) and (3), with f and g continuously differentiable. Suppose this problem has an interior continuous solution $\hat{y}(t) \in \mathrm{Int}\mathcal{Y}(t)$ with state variable $\hat{x}(t)$. Then there exists a continuously differentiable function $\lambda(t)$ such that $\hat{y}(t)$ and $\hat{x}(t)$ satisfy the first order necessary conditions (FOCH): $x(0) = x_0$,(i)

$$H_y(t, \hat{x}(t), \hat{y}(t), \lambda(t)) = 0 \text{ for all } t \in [0, t_1], \tag{13}$$

$$\dot{\lambda}(t) = -H_x(t, \hat{x}(t), \hat{y}(t), \lambda(t)) \text{ for all } t \in [0, t_1], \tag{14}$$

$$\dot{x}(t) = H_\lambda(t, \hat{x}(t), \hat{y}(t), \lambda(t)) \text{ for all } t \in [0, t_1], \tag{15}$$

and $\lambda(t_1) = 0$, with the Hamiltonian $H(t, x, y, \lambda)$ given by (12).

The Hamiltonian and the Maximum Principle III

Theorem (Necessary Cond. II cont.)

Moreover, the Hamiltonian $H(t, x, y, \lambda)$ also satisfies the Maximum Principle that

$$H(t, \hat{x}(t), \hat{y}(t), \lambda(t)) \geq H(t, \hat{x}(t), y, \lambda(t)) \text{ for all } y \in \mathcal{Y}(t),$$

for all $t \in [0, t_1]$.

More generally, FOCH 13 can be replaced by FOCH 13* i.e. $\underset{y}{Max} H$

The Hamiltonian and the Maximum Principle IV

Nice Property: $\lambda(t)$ has an economic interpretation. To see this let

$$
\begin{aligned}
V(t_0, x_0) &= \int_{t_0}^{t_1} f(t, \hat{x}(t), \hat{y}(t))dt \\
&= \int_{t_0}^{t_1} \left[\begin{array}{c} f(t, \hat{x}(t), \hat{y}(t)) + \lambda(t)g(t, \hat{x}(t), \hat{y}(t)) \\ -\lambda(t)\dot{\hat{x}}(t) \end{array} \right] dt \\
&= \int_{t_0}^{t_1} \left[\begin{array}{c} f(t, \hat{x}(t), \hat{y}(t)) + \lambda(t)g(t, \hat{x}(t), \hat{y}(t)) \\ +\dot{\lambda}(t)\hat{x}(t) \end{array} \right] dt \\
&\quad + \lambda(t_0)\hat{x}(t_0) - \lambda(t_1)\hat{x}(t_1)
\end{aligned}
$$

The Hamiltonian and the Maximum Principle V

Fix $\lambda(t)$ and vary x_0 to $x_0 + a$

$$
\begin{aligned}
V(t_0, x_0 + a) &= \int_{t_0}^{t_1} f dt \\
&= \int_{t_0}^{t_1} [f + \lambda g + \dot{\lambda} x] dt + \lambda(t_0)(x_0 + a) - \lambda(t_1) x(t_1)
\end{aligned}
$$

Then

$$
\begin{aligned}
V(t_0, x_0 + a) - V(t_0, x_0) = \int_{t_0}^{t_1} \Big\{ &[\hat{f}_x + \lambda \hat{g}_x + \dot{\lambda}](x - \hat{x}) \\
&+ (\hat{f}_y + \lambda \hat{g}_y)(y - \hat{y}) \Big\} dt \\
&+ \lambda(t_0) a - \lambda(t_1)[x(t_1) - \hat{x}(t_1)] \\
&+ \textit{higher order terms}
\end{aligned}
$$

The Hamiltonian and the Maximum Principle VI

Divide on both sides by a and take the limit as $a \to 0$

Then,

$$\lim_{a \to 0} \frac{V(t_0, x_0 + a) - V(t_0, x_0)}{a} = \frac{\partial V(x_0, t_0)}{\partial x_0} = \lambda(t_0)$$

The costate variable had similar interpretation as multiplier in Lagrangian.

So $\lambda(t)$ gives the value of a marginal increase in $x(t)$ (in period t)

The Hamiltonian and the Maximum Principle VII

- For notational simplicity, in equation (15), $\dot{x}(t)$ instead of $\dot{\hat{x}}(t)$ $(= \partial \hat{x}(t) / \partial t)$.
- Simplified version of the celebrated *Maximum Principle* of Pontryagin:
  1. Find optimal solution by looking jointly for a set of "multipliers" (*costate* variables) $\lambda(t)$ and optimal path of $\hat{y}(t)$ and $\hat{x}(t)$.
  2. $\lambda(t)$ is informative about the value of relaxing the constraint (at time t): value of an infinitesimal increase in $x(t)$ at time t.
  3. $\lambda(t_1) = 0$: after the planning horizon, there is no value to having more x. Finite-horizon equivalent of *transversality condition*.
- Conditions may not be sufficient:
  1. May correspond to stationary points rather than maxima.
  2. May identify a local rather than a global maximum.

The Hamiltonian and the Maximum Principle VIII

Theorem

Consider the problem of maximizing (1) subject to (2) and (3), with f and g continuously differentiable. Define $H(t, x, y, \lambda)$ as in (12), and suppose that an interior continuous solution $\hat{y}(t) \in \text{Int}\mathcal{Y}(t)$ and the corresponding path of state variable $\hat{x}(t)$ satisfy (13)-(15). Suppose also that given the resulting costate variable $\lambda(t)$, $H(t, x, y, \lambda)$ is jointly concave in (x, y) for all $t \in [0, t_1]$, then the $\hat{y}(t)$ and the corresponding $\hat{x}(t)$ achieve a global maximum of (1). Moreover, if $H(t, x, y, \lambda)$ is strictly jointly concave in (x, y) for all $t \in [0, t_1]$, then the pair $(\hat{x}(t), \hat{y}(t))$ achieves the unique global maximum of (1).

The Hamiltonian and the Maximum Principle IX

- Condition that $H(t, x, y, \lambda)$ should be concave is rather demanding.
- Arrow's Theorem weakens these conditions.
- Define the maximized Hamiltonian as

$$M(t, x, \lambda) \equiv \max_{y \in \mathcal{Y}(t)} H(t, x, y, \lambda), \tag{16}$$

  with $H(t, x, y, \lambda)$ itself defined as in (12).
- Clearly, the necessary conditions for an interior maximum in (16) is (13).
- Therefore, if an interior pair of state and control variables $(\hat{x}(t), \hat{y}(t))$ satisfies (13)-(15), then $M(t, \hat{x}, \lambda) \equiv H(t, \hat{x}, \hat{y}, \lambda)$.

The Hamiltonian and the Maximum Principle X

Theorem

Given $(t, \hat{x}(t), \hat{y}(t), \lambda(t))$ *that satisfy the FOCH if* $M(t, \hat{x}(t), \lambda(t))$ *is concave in* $x(t)$ *with* $\mathcal{X}(t)$ *convex for all* $t \in [0, t_1]$, *then* $(\hat{x}(t), \hat{y}(t))$ *is a solution to the problem (1) subject to (2) and (3). Under strict concavity the solution is unique where maximized Hamiltonian*

$$M\left(t, \hat{x}(t), \lambda(t)\right) = \max_{y \in \mathcal{Y}(t)} H\left(t, \hat{x}(t), y(t), \lambda(t)\right)$$

The Hamiltonian and the Maximum Principle XI

Strategy:

1. Find possible (candidate) solutions by 'solving' FOCH $\Longrightarrow (\hat{x}(t), \hat{y}(t), \lambda(t))$
2. Given $(\hat{x}(t), \lambda(t))$ find $M(t, \hat{x}(t), \lambda(t))$and verify concavity

Proposition: If a function $\phi(x, y)$ is concave in (x, y), then $\Phi(X) = \max_y \phi(x, y)$ is also concave.

2'. Given $\lambda(t)$ check that $H(t, x, y, \lambda(t))$ is concave in (x, y)

Proof of Theorem: Arrow's Sufficient Conditions I

- Consider the pair of state and control variables $(\hat{x}(t), \hat{y}(t))$ that satisfy the necessary conditions (13)-(15) as well as (2) and (3).
- Consider also an arbitrary pair $(x(t), y(t))$ that satisfy (2) and (3) and define $M(t, x, \lambda) \equiv \max_y H(t, x, y, \lambda)$.
- Since f and g are differentiable, H and M are also differentiable in x.
- Denote the derivative of M with respect to x by M_x.

Proof of Theorem: Arrow's Sufficient Conditions II

- Then concavity implies that for all $t \in [0, t_1]$.

$$\begin{aligned} M(t, x(t), \lambda(t)) &\leq M(t, \hat{x}(t), \lambda(t)) \\ &\quad + M_x(t, \hat{x}(t), \lambda(t))(x(t) - \hat{x}(t)) \end{aligned}$$

- Integrating both sides over $[0, t_1]$ yields

$$\begin{aligned} \int_0^{t_1} M(t, x(t), \lambda(t))\, dt &\leq \int_0^{t_1} M(t, \hat{x}(t), \lambda(t))\, dt \\ &+ \int_0^{t_1} M_x(t, \hat{x}(t), \lambda(t))(x(t) - \hat{x}(t))\, dt. \end{aligned} \tag{17}$$

Proof of Theorem: Arrow's Sufficient Conditions III

- Moreover, we have

$$\begin{aligned} M_x\left(t, \hat{x}\left(t\right), \lambda\left(t\right)\right) &= H_x\left(t, \hat{x}\left(t\right), \hat{y}\left(t\right), \lambda\left(t\right)\right) \\ &= -\dot{\lambda}\left(t\right), \end{aligned} \tag{18}$$

- First line follows by an Envelope Theorem type reasoning (since $H_y = 0$ from equation (13)), while the second line follows from (14).

Proof of Theorem: Arrow's Sufficient Conditions IV

- Next, exploiting the definition of the maximized Hamiltonian, we have

$$\begin{aligned}\int_0^{t_1} M(t, x(t), \lambda(t))\, dt &= W(x(t), y(t)) \\ &\quad + \int_0^{t_1} \lambda(t)\, g(t, x(t), y(t))\, dt,\end{aligned}$$

and

$$\begin{aligned}\int_0^{t_1} M(t, \hat{x}(t), \lambda(t))\, dt &= W(\hat{x}(t), \hat{y}(t)) \\ &\quad + \int_0^{t_1} \lambda(t)\, g(t, \hat{x}(t), \hat{y}(t))\, dt.\end{aligned}$$

Proof of Theorem: Arrow's Sufficient Conditions V

- Equation (17) together with (18) then implies

$$\begin{aligned} W(x(t), y(t)) \leq \; & W(\hat{x}(t), \hat{y}(t)) \\ & + \int_0^{t_1} \lambda(t) \begin{bmatrix} g(t, \hat{x}(t), \hat{y}(t)) \\ -g(t, x(t), y(t)) \end{bmatrix} dt \\ & - \int_0^{t_1} \dot{\lambda}(t) (x(t) - \hat{x}(t)) \, dt. \end{aligned} \tag{19}$$

Proof of Theorem: Arrow's Sufficient Conditions VI

- Integrating the last term by parts and using the fact that by feasibility $x(0) = \hat{x}(0) = x_0$ and by the transversality condition $\lambda(t_1) = 0$, we obtain

$$\int_0^{t_1} \dot{\lambda}(t)(x(t) - \hat{x}(t))\,dt = -\int_0^{t_1} \lambda(t)\left(\dot{x}(t) - \dot{\hat{x}}(t)\right)dt.$$

- Substituting this into (19), we obtain

$$\begin{aligned} W(x(t), y(t)) \leq\; & W(\hat{x}(t), \hat{y}(t)) \\ & + \int_0^{t_1} \lambda(t)\left[\begin{array}{c} g(t, \hat{x}(t), \hat{y}(t)) \\ -g(t, x(t), y(t)) \end{array}\right]dt \\ & + \int_0^{t_1} \lambda(t)\left[\dot{x}(t) - \dot{\hat{x}}(t)\right]dt. \end{aligned} \tag{20}$$

Proof of Theorem: Arrow's Sufficient Conditions VII

- Since by definition of the admissible pairs $(x(t), y(t))$ and $(\hat{x}(t), \hat{y}(t))$, we have $\dot{\hat{x}}(t) = g(t, \hat{x}(t), \hat{y}(t))$ and $\dot{x}(t) = g(t, x(t), y(t))$, (20) implies that $W(x(t), y(t)) \leq W(\hat{x}(t), \hat{y}(t))$ for any admissible pair $(x(t), y(t))$, establishing the first part of the theorem.
- If M is strictly concave in x, then the inequality in (17) is strict, and therefore the same argument establishes $W(x(t), y(t)) < W(\hat{x}(t), \hat{y}(t))$, and no other $\hat{x}(t)$ could achieve the same value, establishing the second part.

The Hamiltonian and the Maximum Principle XII

- Mangasarian and Arrow Theorems play an important role in the applications of optimal control.
- But are not straightforward to check since neither concavity nor convexity of the $g(\cdot)$ function would guarantee the concavity of the Hamiltonian unless we know something about the sign of the costate variable $\lambda(t)$.
- In many economically interesting situations, we can ascertain $\lambda(t)$ is everywhere positive.
- $\lambda(t)$ is related to the value of relaxing the constraint on the maximization problems; gives another way of ascertaining that it is positive (or negative).
- Then checking Mangasarian conditions is straightforward, especially when f and g are concave functions.

Section 4

Infinite-Horizon Optimal Control

Infinite-Horizon Optimal Control I

- Limitations of above:
  1. We have assumed that a continuous and interior solution to the optimal control problem exists.
  2. So far looked at the finite horizon case, whereas analysis of growth models requires us to solve infinite horizon problems.
  3. Need to look at the more modern theory of optimal control.

Subsection 1

The Basic Problem: Necessary and Sufficient Conditions

The Basic Problem: Necessary and Sufficient Conditions I

- Let us focus on infinite-horizon control with a single control and a single state variable.
- Using the same notation as above, the IHOC problem is

$$\max_{x(t),y(t)} W\left(x\left(t\right), y\left(t\right)\right) \equiv \int_0^\infty f\left(t, x\left(t\right), y\left(t\right)\right) dt \tag{21}$$

subject to

$$\dot{x}\left(t\right) = g\left(t, x\left(t\right), y\left(t\right)\right), \tag{22}$$

and

$$y\left(t\right) \in \mathbb{R} \text{ for all } t,\ x\left(0\right) = x_0 \text{ and } \lim_{t\to\infty} x\left(t\right) \geq x_1. \tag{23}$$

- Allows for an implicit choice over the endpoint x_1, since there is no terminal date.
- The last part of (23) imposes a lower bound on this endpoint.

The Basic Problem: Necessary and Sufficient Conditions II

- Further simplified by removing feasibility requirement that $y(t)$ should always belong to the set $\mathcal{Y}$, instead simply require to be real-valued.
- Have not assumed that the state variable $x(t)$ lies in a compact set.
- Call a pair $(x(t), y(t))$ *admissible* if $y(t)$ is Lebesgue-measurable and thus $x(t)$ is absolutely continuous.
- Define the *value function*, analog of discrete time dynamic programming:

$$
\begin{aligned}
V(t_0, x_0) &\equiv \max_{x(t)\in\mathbb{R}, y(t)\in\mathbb{R}} \int_{t_0}^{\infty} f(t, x(t), y(t))\, dt \qquad (24)\\
\text{subject to } \dot{x}(t) &= g(t, x(t), y(t)),\ x(t_0) = x_0\\
\text{and } \lim_{t\to\infty} b(t)x(t) &\geq x_1\\
y(t) \in \mathbb{R} \text{ for all } t \quad &\text{and} \quad \lim_{t\to\infty} b(t) < \infty \qquad (25)
\end{aligned}
$$

The Basic Problem: Necessary and Sufficient Conditions III

- $V(t_0, x_0)$ gives the optimal value of the dynamic maximization problem starting at time t_0 with state variable x_0.
- Clearly,

$$V(t_0, x_0) \geq \int_{t_0}^{\infty} f(t, x(t), y(t))\, dt \quad \text{for any admissible pair } (x(t), y(t)). \tag{26}$$

- When "max" is not reached, we should be using "sup" instead.
- But we have assumed that all admissible pairs give finite value, so that $V(t_0, x_0) < \infty$, and our focus throughout will be on admissible pairs $(\hat{x}(t), \hat{y}(t))$ that are optimal solutions to (21) subject to (22) and (23), and thus reach the value $V(t_0, x_0)$.

The Basic Problem: Necessary and Sufficient Conditions IV

Theorem (Infinite-Horizon Maximum Principle)

Suppose that the problem of maximizing (21) subject to (22) and (23), with f and g continuously differentiable, has an interior continuous solution $\hat{y}(t)$ with corresponding path of state variable $\hat{x}(t)$. Let $H(t, x, y, \lambda)$ be given by (12). Then the optimal control $\hat{y}(t)$ and the corresponding path of the state variable $\hat{x}(t)$ are such that the Hamiltonian $H(t, x, y, \lambda)$ satisfies the Maximum Principle, that

$$H(t, \hat{x}(t), \hat{y}(t), \lambda(t)) \geq H(t, \hat{x}(t), y, \lambda(t)) \quad \text{for all } y(t), \text{for all } t \in \mathbb{R}$$

The Basic Problem: Necessary and Sufficient Conditions V

Theorem

Moreover, the following FOCIHOC necessary conditions are satisfied:

$$H_y\left(t, \hat{x}\left(t\right), \hat{y}\left(t\right), \lambda\left(t\right)\right) = 0, \tag{27}$$

$$\dot{\lambda}\left(t\right) = -H_x\left(t, \hat{x}\left(t\right), \hat{y}\left(t\right), \lambda\left(t\right)\right), \tag{28}$$

$$\dot{x}\left(t\right) = H_\lambda\left(t, \hat{x}\left(t\right), \hat{y}\left(t\right), \lambda\left(t\right)\right), \textit{ with } x\left(0\right) = x_0 \tag{29}$$

$$\textit{and } \lim_{t\to\infty} \hat{x}\left(t\right) \geq x_{1,} \ \lim_{t\to\infty} b(t)\hat{x}(t) \geq x_{1,} \tag{30}$$

for all $t \in \mathbb{R}_+$.

The Basic Problem: Necessary and Sufficient Conditions VI

- Notice that whenever an interior continuous optimal solution of the specified form exists, it satisfies the Maximum Principle.
- Conditions can be generalized to piecewise continuous functions and non-interior functions (see Kamien and Schwartz (1991) or Ch.4 in Handbook of Mathematical Economics)
- The boundary condition $\lim_{t\to\infty} x(t) \geq x_1$ can be generalized to $\lim_{t\to\infty} b(t)\, x(t) \geq x_1$ for some positive function $b(t)$.
- Sufficient conditions to ensure that such a solution exist are somewhat involved
- In addition, if the optimal control, $\hat{y}(t)$, is a continuous function of time, the conditions (27)-(29) are also satisfied.

The Basic Problem: Necessary and Sufficient Conditions VII

- Most generally, $\hat{y}(t)$ is a Lebesgue measurable function (so it could have discontinuities).
- Added generality of allowing discontinuities is somewhat superfluous in most economic applications
- In most economic problems sufficient to focus on the necessary conditions (27)-(29).

Hamilton-Jacobi-Bellman Equations

- Necessary conditions can also be expressed in the form of the so-called *Hamilton-Jacobi-Bellman* (HJB) equation.

Theorem (Hamilton-Jacobi-Bellman Equations)

Let $V(t, x)$ be as defined in (24) and suppose that the hypotheses in the Infinite-Horizon Maximum Principle Theorem hold. Then whenever $V(t, x)$ is differentiable in (t, x), the optimal pair $(\hat{x}(t), \hat{y}(t))$ satisfies the HJB equation. For all $t \in \mathbb{R}$.

$$\begin{aligned} 0 &= f(t, \hat{x}(t), \hat{y}(t)) + \frac{\partial V(t, \hat{x}(t))}{\partial t} \\ &\quad + \frac{\partial V(t, \hat{x}(t))}{\partial x} g(t, \hat{x}(t), \hat{y}(t)) \end{aligned} \tag{31}$$

The Basic Problem: Necessary and Sufficient Conditions VIII

- **Proof:**
  - The continuous time version of Bellman's Principle of Optimality implies that for the optimal pair $(\hat{x}(t), \hat{y}(t))$,

$$V(t_0, x_0) = \int_{t_0}^{t} f(s, \hat{x}(s), \hat{y}(s))\, ds + V(t, \hat{x}(t)) \text{ for all } t.$$

  - Differentiating this with respect to t and using the differentiability of V and Leibniz's rule,

$$f(t, \hat{x}(t), \hat{y}(t)) + \frac{\partial V(t, \hat{x}(t))}{\partial t} + \frac{\partial V(t, \hat{x}(t))}{\partial x}\dot{x}(t) = 0 \text{ for all } t.$$

  Setting $\dot{x}(t) = g(t, \hat{x}(t), \hat{y}(t))$ gives (31).

The Basic Problem: Necessary and Sufficient Conditions IX

- Note important features:
  1. Given that the continuous differentiability of f and g, the assumption that $V(t, x)$ is differentiable is not very restrictive, since the optimal control $\hat{y}(t)$ is piecewise continuous.
     - From the definition (24), at all t where $\hat{y}(t)$ is continuous, $V(t, x)$ will also be differentiable in t.
     - Moreover, an envelope theorem type argument also implies that when $\hat{y}(t)$ is continuous, $V(t, x)$ should also be differentiable in x
  2. (31) is a partial differential equation, since it features the derivative of V with respect to both time t and the state variable x (Not always easy to solve...analytically).

The Basic Problem: Necessary and Sufficient Conditions X

- Since in this Theorem there is no boundary, may expect that there should be a transversality condition similar to the condition that $\lambda(t_1) = 0$
- Might be tempted to impose a transversality condition of the form

$$\lim_{t \to \infty} \lambda(t) = 0, \tag{32}$$

- But this is not in general the case. A milder transversality condition of the form

$$\lim_{t \to \infty} H(t, x, y, \lambda) = 0 \tag{33}$$

always applies, but is not easy to check.

The Basic Problem: Necessary and Sufficient Conditions XI

Theorem (Arrow's Sufficient Conditions for Infinite Horizon)

Consider the problem of maximizing (21) subject to (22) and (23), with f and g continuously differentiable. Define $H(t, x, y, \lambda)$ as in (12), and suppose that a piecewise continuous solution $\hat{y}(t)$ and the corresponding path of state variable $\hat{x}(t)$ satisfy (27)-(29). Given the resulting costate variable $\lambda(t)$, define $M(t, x, \lambda) \equiv \max_{y(t)\in\mathcal{Y}(t)} H(t, x, y, \lambda)$. If $\mathcal{X}(t)$ is convex for all t, $M(t, x, \lambda)$ is concave in x, and $\lim_{t\to\infty} \lambda(t)(\hat{x}(t) - \tilde{x}(t)) \leq 0$ (TVC) for all $\tilde{x}(t)$ implied by an admissible control path $\tilde{y}(t)$, then the pair $(\hat{x}(t), \hat{y}(t))$ achieves the unique global maximum of (21). Under strict concavity $(\hat{x}, \hat{y})$ is the unique solution.

The Basic Problem: Necessary and Sufficient Conditions XII

- Since $x(t)$ can potentially grow without bounds and we require only concavity (not strict concavity), can apply to models with constant returns and endogenous growth
- Both involve the difficulty to check condition that $\lim_{t\to\infty}\lambda(t)(x(t)-\tilde{x}(t))\leq 0$ for all $\tilde{x}(t)$ implied by an admissible control path $\tilde{y}(t)$.
- This condition will disappear when we can impose a proper transversality condition.

The Basic Problem: Necessary and Sufficient Conditions XIII

- More general TVC

$$\lim_{t\to\infty} H(t, \hat{x}(t), \hat{y}(t), \lambda(t)) = 0$$

Theorem

Let $(\hat{x}(t), \hat{y}(t))$ be a piecewise continuous solution of IHOC. Assume $V(t, x(t))$ is differentiable in $x(t)$ and t (for large t) and the $\lim_{t\to\infty} \partial V(t, \hat{x}(t))/\partial t = 0$, then the solution $(\hat{x}(t), \hat{y}(t))$ satisfies FOCIHOC for some continuously differentiable $\lambda(t)$ and $\lim_{t\to\infty} H(t, \hat{x}, \hat{y}, \lambda) = 0$

Subsection 2

Economic Intuition

Economic Intuition I

FOCIHOC 1:

- Consider the problem of maximizing

$$\int_0^{t_1} H\left(t, \hat{x}(t), y(t), \lambda(t)\right) dt = \int_0^{t_1} \left[\begin{array}{c} f\left(t, \hat{x}(t), y(t)\right) + \\ \lambda(t) g\left(t, \hat{x}(t), y(t)\right) \end{array} \right] dt \tag{34}$$

  with respect to the entire function $y(t)$ for given $\lambda(t)$ and $\hat{x}(t)$, where t_1 can be finite or equal to $+\infty$.
- The condition $H_y\left(t, \hat{x}(t), y(t), \lambda(t)\right) = 0$ would then be a necessary condition for this alternative maximization problem.
- Therefore, the Maximum Principle is implicitly maximizing the sum the original maximand $\int_0^{t_1} f\left(t, \hat{x}(t), y(t)\right) dt$ plus an additional term $\int_0^{t_1} \lambda(t) g\left(t, \hat{x}(t), y(t)\right) dt$.

Economic Intuition II

- Let $V(t, \hat{x}(t))$ be the value of starting at time t with state variable $\hat{x}(t)$ and pursuing the optimal policy from then on.
- We will see that
$$\lambda(t) = \frac{\partial V(t, \hat{x}(t))}{\partial x}.$$
- Consequently, $\lambda(t)$ is the (shadow) value of relaxing the constraint (22) by increasing the value of $x(t)$ at time t.
- Moreover, recall that $\dot{x}(t) = g(t, \hat{x}(t), y(t))$, so that the second term in the Hamiltonian is equivalent to $\int_0^{t_1} \lambda(t) \dot{x}(t) \, dt$.

Economic Intuition III

- This is clearly the shadow value of $x(t)$ at time t and the increase in the stock of $x(t)$ at this point.
- Can think of $x(t)$ as a "stock" variable in contrast to the control $y(t)$, which corresponds to a "flow" variable.
- Therefore, maximizing (34) is equivalent to maximizing instantaneous returns as given by the function $f(t, \hat{x}(t), y(t))$, plus the value of stock of $x(t)$, as given by $\lambda(t)$, times the increase in the stock, $\dot{x}(t)$.
- Thus essence of the Maximum Principle is to maximize the flow return plus the value of the current stock of the state variable.

Economic Intuition IV FOCIHOC 2 (Costate equation)

- Let us now turn to the interpreting the costate equation,

$$\begin{aligned} \dot{\lambda}(t) &= -H_x(t, \hat{x}(t), \hat{y}(t), \lambda(t)) \\ &= -f_x(t, \hat{x}(t), \hat{y}(t)) - \lambda(t) g_x(t, \hat{x}(t), \hat{y}(t)). \end{aligned}$$

- Since $\lambda(t)$ is the value of the stock of the state variable, $x(t)$, $\dot{\lambda}(t)$ is the appreciation in this stock variable.
- A small increase in x will change the current flow return plus the value of the stock by H_x, and the value of the stock by the amount $\dot{\lambda}(t)$.
- This gain should be equal to the depreciation in the value of the stock, $-\dot{\lambda}(t)$.
- Otherwise, it would be possible to change the $x(t)$ and increase the value of $\int_0^\infty H(t, x(t), y(t))\, dt$.

Economic Intuition V Maximum Principle (again)

- Second and complementary intuition for the Maximum Principle comes from the HJB equation (31).
- Consider an exponentially discounted problem, $f(t, x(t), y(t)) = \exp(-\rho t) f(x(t), y(t))$.
- Law of motion of the state variable given by an autonomous differential equation, i.e., $g(t, x(t), y(t)) = g(x(t), y(t))$.
- In this case:
  1. if an admissible pair $(\hat{x}(t), \hat{y}(t))_{t \geq 0}$ is optimal starting at $t = 0$ with initial condition $x(0) = x_0$, it is also optimal starting at $s > 0$, starting with the same initial condition,
  2. that is, $(\hat{x}(t), \hat{y}(t))_{t \geq s}$ is optimal for the problem with initial condition $x(s) = x_0$.
- In view of this, define $V(x) \equiv V(0, x)$, value of pursuing the optimal plan $(\hat{x}(t), \hat{y}(t))$ starting with initial condition x, evaluated at $t = 0$.

Economic Intuition VI

- Since $(\hat{x}(t), \hat{y}(t))$ is an optimal plan irrespective of the starting date, we have that $V(t, x(t)) \equiv \exp(-\rho t) V(x(t))$.
- Then, by definition,

$$\frac{\partial V(t, x(t))}{\partial t} = -\exp(-\rho t) \rho V(x(t)).$$

- Moreover, let $\dot{V}(x(t)) \equiv (\partial V(t, x(t)) / \partial x) \dot{x}(t)$.
- Substituting these expressions into (31) and noting that $\dot{x}(t) = g(\hat{x}(t), \hat{y}(t))$, we obtain the "stationary" form of the Hamilton-Jacobi-Bellman

$$\rho V(x(t)) = f(\hat{x}(t), \hat{y}(t)) + \dot{V}(x(t)). \tag{35}$$

- Can be interpreted as a "no-arbitrage asset value equation"
- Think of V as the value of an asset traded in the stock market and ρ as the required rate of return for (a large number of) investors.

Economic Intuition VI

- Return on the assets come from two sources.
- First, "dividends," the flow payoff $f(\hat{x}(t), \hat{y}(t))$.
- If this dividend were constant and equal to d, and there were no other returns, $V = d/\rho$ or
$$\rho V = d.$$
- Returns also come from capital gains or losses (appreciation or depreciation of the asset), $\dot{V}$.
- Therefore, instead of $\rho V = d$, we have
$$\rho V(x(t)) = d + \dot{V}(x(t)).$$
- Thus Maximum Principle amounts to requiring that $V(x(t))$ and $\dot{V}(x(t))$, should be consistent with this no-arbitrage condition.

Section 5

More on Transversality Conditions

More on Transversality Conditions: Counterexample I

- Consider the following problem:

$$\max \int_0^{\infty} \left[\log\left(c\left(t\right)\right) - \log c^*\right] dt$$

subject to

$$\dot{k}\left(t\right) = \left[k\left(t\right)\right]^{\alpha} - c\left(t\right) - \delta k\left(t\right)$$
$$k\left(0\right) = 1$$

and

$$\lim_{t\to\infty} k\left(t\right) \geq 0$$

where $c^* \equiv \left[k^*\right]^{\alpha} - \delta k^*$ and $k^* \equiv \left(\alpha/\delta\right)^{1/(1-\alpha)}$.

- c^* is the maximum level of consumption that can be achieved in steady state.
- k^* is the corresponding steady-state level of capital.

More on Transversality Conditions: Counterexample II

- The integral converges and takes a finite value (since $c(t)$ cannot exceed c^* forever).
- Hamiltonian,

$$H(k, c, \lambda) = [\log c(t) - \log c^*] + \lambda \left[k(t)^{\alpha} - c(t) - \delta k(t)\right],$$

- Necessary conditions (dropping time dependence):

$$\begin{aligned} H_c(k, c, \lambda) &= \frac{1}{c(t)} - \lambda(t) = 0 \\ H_k(k, c, \lambda) &= \lambda(t)\left(\alpha k(t)^{\alpha-1} - \delta\right) = -\dot{\lambda}(t). \end{aligned}$$

More on Transversality Conditions: Counterexample III

- Any optimal path must feature $c(t) \to c^*$ as $t \to \infty$. This, however, implies
$$\lim_{t\to\infty} \lambda(t) = \frac{1}{c^*} > 0 \text{ and } \lim_{t\to\infty} k(t) = k^*.$$
- Thus the equivalent of the standard finite-horizon transversality conditions do not hold.
- It can be verified, however, that along the optimal path we have
$$\lim_{t\to\infty} H(k(t), c(t), \lambda(t)) = 0.$$

More on Transversality Conditions

Theorem

Suppose that problem of maximizing (21) subject to (22) and (23), with f *and* g *continuously differentiable, has an interior piecewise continuous solution* $\hat{y}(t)$ *with corresponding path of state variable* $\hat{x}(t)$*. Suppose moreover that* $V(t, \hat{x}(t))$ *is differentiable in* x *and* t *for* t *sufficiently large and that* $\lim_{t\to\infty} \partial V(t, \hat{x}(t)) / \partial t = 0$*. Let* $H(t, x, y, \lambda)$ *be given by (12). Then the optimal control* $\hat{y}(t)$ *and the corresponding path of the state variable* $\hat{x}(t)$ *satisfy the necessary conditions (27)-(29) and the transversality condition*

$$\lim_{t\to\infty} H(t, \hat{x}(t), \hat{y}(t), \lambda(t)) = 0. \tag{36}$$

Proof of Theorem: Transversality Conditions for Infinite-Horizon Problems

- Focus on points where $V(t, x)$ is differentiable in t and x so that the Hamilton-Jacobi-Bellman equation, (31) holds.
- Noting that $\partial V(t, \hat{x}(t)) / \partial x = \lambda(t)$, this equation can be written as, for all t

$$\begin{aligned} \frac{\partial V(t, \hat{x}(t))}{\partial t} + f(t, \hat{x}(t), \hat{y}(t)) + \lambda(t) g(t, \hat{x}(t), \hat{y}(t)) &= 0 \\ \frac{\partial V(t, \hat{x}(t))}{\partial t} + H(t, \hat{x}(t), \hat{y}(t), \lambda(t)) &= 0. \end{aligned}$$

- Now take the limit as $t \rightarrow \infty$ and use $\lim_{t\rightarrow\infty} \partial V(t, \hat{x}(t)) / \partial t = 0$ to obtain (36).

Section 6

Discounted Infinite-Horizon Optimal Control

Discounted Infinite-Horizon Optimal Control (DIHOC) I

- Part of the difficulty, especially for transversality condition, comes from not enough structure on f and g.
- Economically interesting problems often take the following more specific form:

$$\max_{x(t),y(t)} W(x(t), y(t)) \equiv \int_0^\infty \exp(-\rho t) f(x(t), y(t))\, dt \text{ with } \rho > 0, \tag{37}$$

subject to

$$\dot{x}(t) = g(x(t), y(t)), \tag{38}$$

and

$$y(t) \in \mathbb{R} \text{ for all } t,\ x(0) = x_0 \text{ and } \lim_{t\to\infty} b(t)x(t) \geq x_1, \tag{39}$$

$$b : \mathbb{R}_+ \to \mathbb{R}_+,\ \lim_{t\to\infty} b(t) < \infty \tag{40}$$

- Assume $\rho > 0$, so that there is indeed *discounting*.
- Key: f depends on time only through exponential discounting, g does not depend directly on t.

Discounted Infinite-Horizon Optimal Control (DIHOC) II

- Hamiltonian:

$$\begin{aligned} H\left(t, x\left(t\right), y\left(t\right), \lambda\left(t\right)\right) &= \exp\left(-\rho t\right) f\left(x\left(t\right), y\left(t\right)\right) + \lambda\left(t\right) g\left(x\left(t\right), y\left(t\right)\right) \\ &= \exp\left(-\rho t\right)\left[f\left(x\left(t\right), y\left(t\right)\right) + \mu\left(t\right) g\left(x\left(t\right), y\left(t\right)\right)\right], \end{aligned}$$

where the second line defines

$$\mu\left(t\right) \equiv \exp\left(\rho t\right) \lambda\left(t\right). \tag{41}$$

$$\Longleftrightarrow \lambda(t) = \mu(t) e^{-\rho t} \tag{42}$$

- System is "autonomous", since the Hamiltonian depends on time explicitly only through the $\exp\left(-\rho t\right)$ term.

Discounted Infinite-Horizon Optimal Control (DIHOC) III

- In fact can work with *the current-value Hamiltonian*,

$$\hat{H}\left(t, x\left(t\right), y\left(t\right), \mu\left(t\right)\right) \equiv f\left(x\left(t\right), y\left(t\right)\right) + \mu\left(t\right) g\left(x\left(t\right), y\left(t\right)\right) = e^{\rho t} H \tag{43}$$

- "Autonomous" problems are such that they do not depend directly on time.
- Refer to $f\left(x, y\right)$ and $g\left(x, y\right)$ as weakly monotone if each one is monotone in each of its arguments.
- Assume the optimal control $\hat{y}\left(t\right)$ is everywhere a continuous function of time, for simplicity.

Discounted Infinite-Horizon Optimal Control IV

Assumption (7.1)

Let

- $f(.)$ *be weakly monotone in* (x, y); $g(.)$ *be weakly monotone in* (t, x, y)
- g_y *be bounded away from zero i.e.* $\exists m > 0$ *s.t.* $|g_y(t, x, y)| \geq m$
- f_y *be bounded i.e.* $|f_y(x, y)| \leq M$, $0 < M < \infty$

Maximum Principle for Discounted Infinite-Horizon Problems (DIHOC) I

Theorem

Suppose that problem (37) subject to (38) and (39), with f and g continuously differentiable has an interior piecewise continuous optimal control $\hat{y}(t)$ with corresponding state variable $\hat{x}(t)$. Let $V(t, x(t))$ be the Value function 24. Assume $V(t, x(t))$ is differentiable in t and x for large enough t, that $V(t, \hat{x}(t)) < \infty$ for all t, and $\lim_{t\to\infty} \partial V(t, \hat{x}(t))/\partial t = 0$. Let $\hat{H}(t, \hat{x}, \hat{y}, \mu)$ be the current-value Hamiltonian given by (43). Then except at points of discontinuity the optimal control $\hat{y}(t)$ and the corresponding path of the state variable $\hat{x}(t)$ satisfy the following necessary conditions:

$$\hat{H}_y(t, \hat{x}(t), \hat{y}(t), \mu(t)) = 0 \text{ for all } t \in \mathbb{R}_+, \tag{44}$$

$$\rho\mu(t) - \dot{\mu}(t) = \hat{H}_x(t, \hat{x}(t), \hat{y}(t), \mu(t)) \text{ for all } t \in \mathbb{R}_+, \tag{45}$$

$$\dot{x}(t) = \hat{H}_\mu(t, \hat{x}(t), \hat{y}(t), \mu(t)) \text{ for all } t \in \mathbb{R}_+, \tag{46}$$

$$x(0) = x_0 \text{ and } \lim_{t\to\infty} x(t) \geq x_1,$$

Maximum Principle for Discounted Infinite-Horizon Problems (DIHOC) II

Theorem (cont.)

and the transversality condition

$$\lim_{t\to\infty} \exp\left(-\rho t\right) \hat{H}\left(\hat{x}\left(t\right), \hat{y}\left(t\right), \mu\left(t\right)\right) = 0. \tag{47}$$

Moreover, if Assumption 7.1 holds and *either*
$\lim_{t\to\infty} \hat{x}(t) = x^*$,
$\lim_{t\to\infty} \dot{x}(t)/\hat{x}(t) = \chi$,
then the transversality condition can be strengthened to:

$$\lim_{t\to\infty} \left[\exp\left(-\rho t\right) \mu\left(t\right) \hat{x}\left(t\right)\right] = 0. \tag{48}$$

Discounted Infinite-Horizon Optimal Control V

- The condition
$$\lim_{t\to\infty}\left[\exp\left(-\rho t\right)\mu\left(t\right)\hat{x}\left(t\right)\right]=0$$
is the transversality condition used in most economic applications.
- However, as the previous theorem shows, it holds only under additional assumptions.
- Moreover, only for interior piecewise continuous solutions.
- Again: does such a solution exist?
- It turns out that this question can be answered in most economic problems, because this transversality condition is *sufficient* for concave problems.

Discounted Infinite-Horizon Optimal Control VI

Theorem (Sufficient Conditions for Discounted Infinite-Horizon Problems)

Consider the problem of maximizing (37) subject to (38) and (39), with f and g continuously differentiable. Define $\hat{H}(t, x, y, \mu)$ as the current-value Hamiltonian as in (43), and suppose that a solution $\hat{y}(t)$ and the corresponding path of state variable $\hat{x}(t)$ satisfy (44)-(46) and the stronger transversality condition

$$\lim_{t \to \infty} \left[\exp(-\rho t)\, \mu(t)\, \hat{x}(t)\right] = 0.$$

Assume $V(t, \hat{x}(t)) < \infty$ for all t. Given the resulting current-value costate variable $\mu(t)$, define $M(t, x, \mu) \equiv \max_{y(t) \in \mathcal{Y}(t)} \hat{H}(x, y, \mu)$. Suppose that for any admissible pair $(x(t), y(t))$, $\lim_{t \to \infty} \left[\exp(-\rho t)\, \mu(t)\, x(t)\right] \geq 0$ and that $M(t, x, \mu)$ is concave in x. Then $\hat{y}(t)$ and the corresponding $\hat{x}(t)$ achieve the unique global maximum of (37). If M is strictly concave, then the solution is unique.

General Strategy for Infinite-Horizon Optimal Control Problems

1. Use the necessary conditions given by the Maximum Principle to construct a *candidate solution*
2. Check that this candidate solution satisfies the sufficiency conditions.

- This strategy will work in almost all growth models.

Section 7

Existence of Solutions

Existence of Solutions

- So far, no general result on existence of solutions.
- This can be stated and proved (see book if interested).
- But not so useful for our interests...two reasons:
  1. conditions for existence of interior and continuous solutions much more complicated
  2. the strategy of verifying sufficiency conditions much more straightforward.

Section 8

Examples

Subsection 1

Natural Resource Allocation

Example: Natural Resource I

- Infinitely-lived individual that has access to a non-renewable or exhaustible resource of size 1.
- Instantaneous utility of consuming a flow of resources y is $u(y)$.
- $u : [0,1] \to \mathbb{R}$ is a strictly increasing, continuously differentiable and strictly concave function.
- Objective function at time $t = 0$ is to maximize

$$\int_0^\infty \exp(-\rho t)\, u(y(t))\, dt.$$

- The constraint is that the remaining size of the resource at time t, $x(t)$ evolves according to

$$\dot{x}(t) = -y(t),$$

- Also need that $x(t) \geq 0$.

Example: Natural Resource II

- Current-value Hamiltonian

$$\hat{H}(x(t), y(t), \mu(t)) = u(y(t)) - \mu(t)\, y(t).$$

- Necessary condition for an interior continuously differentiable solution $(\hat{x}(t), \hat{y}(t))$. There should exist a continuously differentiable function $\mu(t)$ such that

$$u'(\hat{y}(t)) = \mu(t),$$

and

$$\dot{\mu}(t) = \rho\mu(t).$$

- The second condition follows since neither the constraint nor the objective function depend on $x(t)$.
- This is the *Hotelling rule* for the exploitation of exhaustible resources.

Example: Natural Resource III

- Integrating both sides of this equation and using the boundary condition,

$$\mu(t) = \mu(0)\exp(\rho t).$$

- Now combining this with the first-order condition for $y(t)$,

$$\hat{y}(t) = u'^{-1}[\mu(0)\exp(\rho t)],$$

- $u'^{-1}[\cdot]$ exists and is strictly decreasing since u is strictly concave.
- Thus amount of the resource consumed is monotonically decreasing over time:
  - because of discounting, preference for early consumption, and delayed consumption has no return.
  - but not all consumed immediately, also a preference for smooth consumption from $u(\cdot)$ is strictly concave.

Example: Natural Resource IV

- Combining the previous equation with the resource constraint,

$$\dot{x}(t) = -u'^{-1}\left[\mu(0)\exp(\rho t)\right].$$

- Integrating this equation and using the boundary condition that $x(0) = 1$,

$$\hat{x}(t) = 1 - \int_0^t u'^{-1}\left[\mu(0)\exp(\rho s)\right] ds.$$

- Since along any optimal path we must have $\lim_{t\to\infty}\hat{x}(t) = 0$,

$$\int_0^\infty u'^{-1}\left[\mu(0)\exp(\rho s)\right] ds = 1.$$

- Therefore, $\mu(0)$ must be chosen so as to satisfy this equation.

Subsection 2

A First Look at Optimal Growth in Continuous Time

A First Look at Optimal Growth in Continuous Time I

- Neoclassical economy without any population growth and without any technological progress.
- Optimal growth problem in continuous time can be written as:

$$\max_{[k(t),c(t)]_{t=0}^{\infty}} \int_0^{\infty} \exp\left(-\rho t\right) u\left(c\left(t\right)\right) dt,$$

  subject to

$$\dot{k}\left(t\right) = f\left(k\left(t\right)\right) - \delta k\left(t\right) - c\left(t\right)$$

  and $k\left(0\right) > 0$.

- $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is strictly increasing, continuously differentiable and strictly concave.
- $f\left(\cdot\right)$ satisfies Assumptions 1 and 2.

A First Look at Optimal Growth in Continuous Time II

- The constraint function, $f(k) - \delta k - c$, is decreasing in c, but may be non-monotone in k.
- But we can restrict attention to $k(t) \in [0, \bar{k}]$, where $\bar{k}$ is defined such that $f'(\bar{k}) = \delta$, so constrained function is also weakly monotone.
- Current-value Hamiltonian,

$$\hat{H}(k, c, \mu) = u(c(t)) + \mu(t)[f(k(t)) - \delta k(t) - c(t)], \tag{49}$$

- Necessary conditions:

$$\begin{aligned}
\hat{H}_c(k, c, \mu) &= u'(c(t)) - \mu(t) = 0 \\
\hat{H}_k(k, c, \mu) &= \mu(t)\left(f'(k(t)) - \delta\right) = \rho\mu(t) - \dot{\mu}(t) \\
\lim_{t\to\infty}[\exp(-\rho t)\mu(t)k(t)] &= 0.
\end{aligned}$$

A First Look at Optimal Growth in Continuous Time III

- First necessary condition immediately implies that $\mu(t) > 0$ (since $u' > 0$ everywhere).
- Thus current-value Hamiltonian is sum of two strictly concave functions and is itself strictly concave.
- Moreover, since $k(t) \geq 0$, for any admissible solution $\lim_{t\to\infty}[\exp(-\rho t)\,\mu(t)\,k(t)] \geq 0$.
- Hence a solution that satisfies these necessary conditions in fact gives a *global maximum.*
- Characterizing the solution of these necessary conditions also establishes the existence of a solution in this case.

Subsection 3

The q-Theory of Investment

The q-Theory of Investment I

- Canonical model of investment under adjustment costs, also known as the q-theory of investment.
- Price-taking firm trying to maximize the present discounted value of its profits.
- Firm is subject to "adjustment" costs when it changes its capital stock.
- Let the capital stock of the firm be $k(t)$.
- Firm has access to a production function $f(k(t))$ that satisfies Assumptions 1 and 2.
- Normalize the price of the output of the firm to 1 in terms of the final good at all dates.
- Adjustment costs captured by $\phi(i)$: strictly increasing, continuously differentiable and strictly convex, and satisfies $\phi(0) = \phi'(0) = 0$.

The q-Theory of Investment II

- In some models, the adjustment cost is taken to be $\phi\left(i/k\right)$ instead of $\phi\left(i\right)$
- Installed capital depreciates at an exponential rate δ.
- Firm maximizes its net present discounted earnings with a discount rate equal to the interest rate r, constant.
- The firm's problem can be written as

$$\max_{k(t),i(t)} \int_0^\infty \exp\left(-rt\right)\left[f\left(k\left(t\right)\right) - i\left(t\right) - \phi\left(i\left(t\right)\right)\right]dt$$

subject to

$$\dot{k}\left(t\right) = i\left(t\right) - \delta k\left(t\right) \tag{50}$$

and $k\left(t\right) \geq 0$, with $k\left(0\right) > 0$ given.

- Clearly, both the objective function and the constraint function are weakly monotone.
- Since ϕ is strictly convex, not optimal to make "large" adjustments.

The q-Theory of Investment III

- Current-value Hamiltonian:

$$\hat{H}(k,i,q) \equiv [f(k(t)) - i(t) - \phi(i(t))] + q(t)[i(t) - \delta k(t)],$$

- Used $q(t)$ instead of the familiar $\mu(t)$ for the costate variable.
- Necessary conditions for this problem are standard (suppressing the "^" to denote the optimal values):

$$\begin{aligned}
\hat{H}_i(k,i,q) &= -1 - \phi'(i(t)) + q(t) = 0 \\
\hat{H}_k(k,i,q) &= f'(k(t)) - \delta q(t) = rq(t) - \dot{q}(t) \\
\lim_{t\to\infty} \exp(-rt)\, q(t)\, k(t) &= 0.
\end{aligned}$$

- First necessary condition implies,

$$q(t) = 1 + \phi'(i(t)) \text{ for all } t. \tag{51}$$

The q-Theory of Investment IV

- Differentiating with respect to time,
$$\dot{q}(t) = \phi''(i(t))\,\dot{i}(t). \tag{52}$$
- Substituting into the second necessary condition, law of motion for investment:
$$\dot{i}(t) = \frac{1}{\phi''(i(t))}\left[(r+\delta)\left(1+\phi'(i(t))\right) - f'(k(t))\right]. \tag{53}$$
- Interesting economic features:
  - As $\phi''(i)$ tends to zero, $\dot{i}(t)$ diverges, meaning that investment jumps to a particular value.
    - I.e., it can be shown that this value is such that the capital stock immediately reaches its state-state value
    - As $\phi''(i)$ tends to zero, $\phi(i)$ becomes linear: adjustment costs increase cost linearly and no need for smoothing.
  - When $\phi''(i(t)) > 0$, smoothing: $\dot{i}(t)$ will take a finite value, and investment will adjust slowly.

The q-Theory of Investment V

- Behavior of investment and capital stock using the differential equations (50) and (53).
- There exists a unique steady-state solution with $k > 0$, and involves $i^* = \delta k^*$.
- This steady-state level of capital satisfies the first-order condition (corresponding to the right-hand side of (53) being equal to zero):

$$f'\left(k^*\right) = \left(r + \delta\right)\left(1 + \phi'\left(\delta k^*\right)\right).$$

- Differs from "modified golden rule:" additional cost means there more investment needed to replenish depreciated capital–term $\phi'\left(\delta k^*\right)$.
- Since ϕ is strictly convex and f is strictly concave and satisfies the Inada conditions, a unique value of k^* satisfies this condition.

The q-Theory of Investment VI

- Instead of global stability in the k-i space, the correct concept here is *saddle-path stability*.
- Instead of an initial value constraint, $i(0)$ is pinned down by a boundary condition at "infinity,"

$$\lim_{t\to\infty} \exp(-rt)\, q(t)\, k(t) = 0.$$

- Thus with one state and one control variable, we should have a one-dimensional manifold (a curve) along which capital-investment pairs tend towards the steady state.
- This manifold is also referred to as the "stable arm".
- $i(0)$ will then be determined so that the economy starts along this manifold.
- If any capital-investment pair were to lead to the steady state, we would not know how to determine $i(0)$; "indeterminacy" of equilibria.

The q-Theory of Investment VII

- Mathematically, saddle-path stability involves the number of negative eigenvalues to be the same as the number of state variables.

Theorem

Consider the following linear differential equation system

$$\dot{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b} \tag{54}$$

with initial value $\mathbf{x}(0)$, *where* $\mathbf{x}(t) \in \mathbb{R}^n$ *for all* t *and* $\mathbf{A}$ *is an* $n \times n$ *matrix. Let* $\mathbf{x}^*$ *be the steady state of the system given by* $\mathbf{A}\mathbf{x}^* + \mathbf{b} = 0$. *Suppose that* $m \leq n$ *of the eigenvalues of* $\mathbf{A}$ *have negative real parts. Then there exists an m-dimensional subspace* M *of* $\mathbb{R}^n$ *such that starting from any* $\mathbf{x}(0) \in M$, *the differential equation (54) has a unique solution with* $\mathbf{x}(t) \to \mathbf{x}^*$.

The q-Theory of Investment VIII

Theorem

Consider the following nonlinear autonomous differential equation

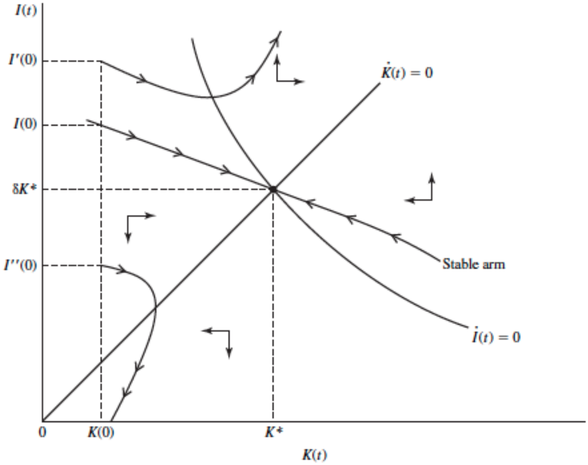
$$\dot{x}(t) = \mathbf{G}(\mathbf{x}(t)) \tag{55}$$

where $\mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n$ *and suppose that* $\mathbf{G}$ *is continuously differentiable, with initial value* $\mathbf{x}(0)$. *Let* $\mathbf{x}^*$ *be a steady-state of this system, given by* $\mathbf{G}(\mathbf{x}^*) = 0$. *Define*

$$\mathbf{A} = D\mathbf{G}(\mathbf{x}^*),$$

and suppose that $m \leq n$ *of the eigenvalues of* $\mathbf{A}$ *have negative real parts and the rest have positive real parts. Then there exists an open neighborhood of* $\mathbf{x}^*$, $\mathbf{B}(\mathbf{x}^*) \subset \mathbb{R}^n$ *and an* m*-dimensional manifold* $M \subset \mathbf{B}(\mathbf{x}^*)$ *such that starting from any* $\mathbf{x}(0) \in M$, *the differential equation (55) has a unique solution with* $\mathbf{x}(t) \to \mathbf{x}^*$.

Dynamics of capital and investment in the q-theory



The q-Theory of Investment IX

- Figure investigates the transitional dynamics in the q-theory of investment.
- Adjustment costs discourage large values of investment: firm cannot adjust to its steady-state level immediately.
- Diminishing returns imply benefit of increasing k is greater when k is low.
- As capital accumulates and $k(t) > k(0)$, the benefit of boosting the capital stock declines and the firm also reduces investment.
- Initial investment $i(0)$ is the unique optimum. Why? *Sufficiency Theorem.*

The q-Theory of Investment X

- Alternative popular approach, use Figure.
- Consider, for example, $i'(0) > i(0)$ as the initial level:
  - $i(t)$ and $k(t)$ would tend to infinity.
  - $q(t)\,k(t)$ would tend to infinity at a rate faster than r, violating the transversality condition, $\lim_{t\to\infty} \exp(-rt)\,q(t)\,k(t) = 0$.
  - Along a trajectory starting at $i'(0)$, $\dot{k}(t)/k(t) > 0$, and thus we have

$$\begin{aligned}
\frac{d(q(t)k(t))/dt}{q(t)k(t)} &\geq \frac{\dot{q}(t)}{q(t)} \\
&= \frac{\dot{i}(t)\phi''(i(t))}{1+\phi'(i(t))} \\
&= r+\delta - f'(k(t))/(1+\phi'(i(t))),
\end{aligned}$$

  - Second line uses (51) and (52), while third line substitutes from (53).
  - As $k(t) \to \infty$, we have that $f'(k(t)) \to 0$, implying that

$$\lim_{t\to\infty} \exp(-rt)\,q(t)\,k(t) \geq \lim_{t\to\infty} \exp(-rt)\exp((r+\delta)\,t) = \lim_{t\to\infty} \exp(\delta t)$$

violating the transversality condition.

The q-Theory of Investment XI

- In contrast, if we start with $i''(0) < i(0)$ as the initial level:
  - $i(t)$ would tend to 0 in finite time
  - $k(t)$ would also tend towards zero (though not reaching it in finite time).
  - After the time where $i(t) = 0$, we also have $q(t) = 1$ and thus $\dot{q}(t) = 0$ (from (51)).
  - Moreover, by the Inada conditions, as $k(t) \to 0$, $f'(k(t)) \to \infty$.
  - Consequently, after $i(t)$ reaches 0, the necessary condition $\dot{q}(t) = (r + \delta) q(t) - f'(k(t))$ is violated (though care necessary, since at the boundary this condition is no longer necessary).

The q-Theory of Investment XII

- "q-theory" aspects (Tobin): value of an extra unit of capital divided by its replacement cost is a measure of the "value of investment".
- When this ratio is high, the firm would like to invest more.
- In steady state, firm will settle where this ratio is 1 or close to 1.
- Costate variable $q(t)$ measures Tobin's q.
- Denote the current (maximized) value of the firm when it starts with a capital stock of $k(t)$ by $V(k(t))$.
- Same arguments as above imply that

$$V'(k(t)) = q(t), \tag{56}$$

- $q(t)$ measures exactly by how much one dollar increase in capital will raise the value of the firm.

The q-Theory of Investment XIII

- In steady state, we have $\dot{q}(t) = 0$, so that $q^* = f'(k^*) / (r + \delta)$, which is approximately equal to 1 when $\phi'(\delta k^*)$ is small.
- Out of steady state, $q(t)$ can be significantly greater than this amount, signaling that there is need for greater investments.
- Tobin's q, or alternatively the costate variable $q(t)$, will play the role of signaling when investment demand is high.
- Proxies for Tobin's q can be constructed using stock market prices and book values of firms.
- When stock market prices are greater than book values, this corresponds a high Tobin's q.
- But whether this is a good approach is intensely debated:
  - Tobin's q does not contain all the relevant information when there are irreversibilities or fixed costs of investment,
  - What is relevant is the "marginal q," but typically only measure "average q".

Section 9

Conclusions

Conclusions

- Basic ideas of optimal control may be a little less familiar than those of discrete time dynamic programming, but used in much of growth theory and in other areas of macroeconomics.
- Moreover, some problems become easier in continuous time.
- The most powerful theorem in optimal control, Pontryagin's Maximum Principle, is as much an economic result as a mathematical result.
- Maximum Principle has a very natural interpretation both in terms of maximizing flow returns plus the value of the stock, and also in terms of an asset value equation for the value of the maximization problem.

Now let's apply these tools to some economic problems