Solow Model

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Macroeconomic Theory II

Section 1

Solow Growth Model

Solow Growth Model

- Develop a simple framework for the *proximate* causes and the mechanics of economic growth and cross-country income differences.
- Solow-Swan model named after Robert (Bob) Solow and Trevor Swan, or simply the *Solow model*
- Before Solow growth model, the most common approach to economic growth built on the Harrod-Domar model.
- Harrod-Domar model emphasized potential dysfunctional aspects of growth: e.g, how growth could go hand-in-hand with increasing unemployment.
- At the center of the Solow growth model is the *neoclassical aggregate* production function.
- Dynamic version of Classical Model

The Economic Environment of the Basic Solow Model

- Study of economic growth and development necessitates dynamic models.
- Despite its simplicity, the Solow growth model is a dynamic general equilibrium model (though many key features of dynamic general equilibrium models, such as preferences and dynamic optimization are missing in this model).

Solow is an algebraic or graphical solution to growth

- One Sector, one good, no government, closed economy no foreign sector
- One representative consumer / household saves $s \in (0, 1)$ of income, consumes $(1 - s)$, performs 1 unit of labor $(L(t))$.
- One representative firm, uses K, L in production.

Households and Production I

- Closed economy, with a unique final good. No foreign sector. One good = one sector.
- Time running to an infinite horizon, time is indexed by $t \in T \subseteq \mathbb{R}_+$.
- Economy is inhabited by a large number of households, and for now households will not be optimizing.
- This is the main difference between the Solow model and the *neoclassical growth model*.
- To fix ideas, assume all households are identical, so the economy admits *a representative household*. One representative household. One representative consumer.
- No government.

Households and Production II

- Assume households save a constant exogenous fraction s of their disposable income
- Same assumption used in basic Keynesian models and in the Harrod-Domar model; at odds with reality.
- Assume all firms have access to the same production function: economy admits a **representative firm**, with a representative (or aggregate) production function. One representative firm.

Households and Production III

- Aggregate production function [P/N F/N] for the unique final good is

$$Y(t) = F[K(t), L(t), A(t)] \tag{1}$$

- Assume capital is the same as the final good of the economy, but used in the production process of more goods.
- $A(t)$ is a *shifter* of the P/N F/N (1). Broad notion of technology.
- Major assumption: technology is **free**; it is publicly available as a non-excludable, non-rival good.

First Key Assumption

Assumption 1 **(Continuity, Twice Continuously Differentiability, Positive and Diminishing Marginal Products, and Constant Returns to Scale)** The production function $F : \mathbb{R}_+^3 \to \mathbb{R}_+$ is twice continuously differentiable in K and L, and satisfies

$$F_K(K, L, A) \equiv \frac{\partial F(\cdot)}{\partial K} > 0, \quad F_L(K, L, A) \equiv \frac{\partial F(\cdot)}{\partial L} > 0,$$
$$F_{KK}(K, L, A) \equiv \frac{\partial^2 F(\cdot)}{\partial K^2} < 0, \quad F_{LL}(K, L, A) \equiv \frac{\partial^2 F(\cdot)}{\partial L^2} < 0.$$

First Key Assumption

Moreover, F exhibits constant returns to scale [CRS] in K and L.

- Assume F exhibits *constant returns to scale* in K and L: $F(\lambda K, \lambda L, A) = \lambda F(K, L, A) \ \forall \ \lambda \in \mathbb{R}_+$. I.e., it is *linearly homogeneous* (homogeneous of degree 1) in these two variables.

Review

Definition (1) Let m be an integer. The function $g : \mathbb{R}^{\alpha+\beta} \to \mathbb{R}$ is homogeneous of degree m in $x \in \mathbb{R}^{\alpha}$ if and only if

$$g(\lambda x, z) = \lambda^{m} g(x, z) \text{ for all } \lambda \in \mathbb{R}_{+} \text{ and } z \in \mathbb{R}^{\beta} \tag{2}$$

Review

Theorem (Euler's Theorem) ($\forall\alpha \in \mathbb{N}$, shown for $\alpha = 2$) Suppose that $g : \mathbb{R}^{\alpha+\beta} \to \mathbb{R}$ is continuously differentiable in $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$, with partial derivatives denoted by g_{x_1} and g_{x_2} and is homogeneous of degree m in x_1 and x_2. Then

$$\begin{aligned} mg\left(x_1, x_2, z\right) &= \nabla_x g(\cdot) \cdot x = g_{x_1}\left(x, z\right) x_1 + g_{x_2}\left(x, z\right) x_2 \\ \text{for all } x &\in \mathbb{R}^{\alpha} \text{ and } z \in \mathbb{R}^{\beta}. \end{aligned}$$

Moreover, $g_{x_1}\left(x_1, x_2, z\right)$ and $g_{x_2}\left(x_1, x_2, z\right)$ are themselves homogeneous of degree $m-1$ in x_1 and x_2.

Second Key Assumption

Assumption 2 **(Inada conditions)** F satisfies the Inada conditions

$$\begin{array}{rcl}\lim_{K\to 0} F_K(\cdot) & = & \infty, \text{ and } \lim_{K\to\infty} F_K(\cdot) = 0 \quad \forall\, A,\ L > 0, \\ \lim_{L\to 0} F_L(\cdot) & = & \infty, \text{ and } \lim_{L\to\infty} F_L(\cdot) = 0 \quad \forall\, A,\ K > 0.\end{array}$$

- Works nicely with intermediate value theorem:
  $\forall\ \gamma \in \mathbb{R}_+$, $\exists$ a unique k such that [s.t.] $F_k(\cdot) = \gamma$
- This can be observed graphically as $F_K(0,\cdot) = \infty$, & $F_K(\infty,\cdot) = 0$
- Important in ensuring the existence of *interior equilibria.*
- It can be relaxed quite a bit, though useful to get us started.

Second Key Assumption

- We assume that inputs and outputs are exchanged in competitive markets.
- A production function is **Neoclassical** if it satisfies Assumptions 1 and 2.
- Note Assumptions 1 and 2: $\rightarrow F(K, 0, A) = F(0, L, A) = 0 \ \forall K, L, A$

Production Functions

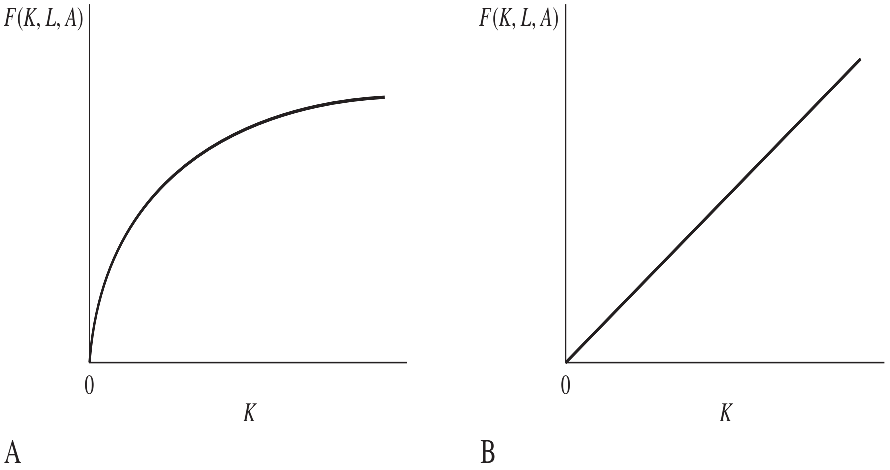


Figure 2.1 – Production functions and the marginal product of capital. (A) satisfies the Inada conditions in Assumption 2, while (B) does not.

Market Structure, Endowments and Market Clearing I

- We will assume that markets are competitive, so ours will be a prototypical *competitive general equilibrium model*.
- Households own all of the labor, which they supply inelastically.
- Endowment of labor in the economy, $\bar{L}(t)$, and all of this will be supplied regardless of the price.
- The *labor market clearing* condition can then be expressed as:

$$L(t) = \bar{L}(t) \tag{3}$$

$\forall\ t$, where $L(t)$ denotes the demand for labor (and also the level of employment). And $\bar{L}(t)$ denotes labor supply.

Market Structure, Endowments and Market Clearing II

- More generally, should be written in complementary slackness form.
- In particular, let the *wage rate* at time t be $w(t)$, then the **labor market clearing** condition takes the form:

$$L(t) \leq \bar{L}(t), \; w(t) \geq 0, \text{ and } \left[L(t) - \bar{L}(t)\right] w(t) = 0 \qquad (4)$$

- But Assumption 1 and competitive labor markets make sure that wages have to be strictly positive.

Market Structure, Endowments and Market Clearing III

- Households own the capital stock of the economy: $\bar{K}(t)$.
- And at time t rents it to the firms at the *rental price of capital:* $R(t)$.
- Capital market clearing condition:

$$K^d(t) = K^s(t) \quad \equiv \quad K(t) = \bar{K}(t) \quad \forall\ \bar{K}(t) \geq 0$$

- Complementary slackness form of **Capital market clearing** condition:

$$K(t) \leq \bar{K}(t),\ R(t) \geq 0,\ \&\ \left[K(t) - \bar{K}(t)\right] R(t) = 0, \quad \forall\ \bar{K}(t) \geq 0$$

- Take households' initial holdings of capital, $\bar{K}(0)$, as given

Section 2

The Solow Model in Discrete Time

Solow Model in Discrete Time

- Our notation until now is such that t could be discrete or continuous (careful book makes no distinction whatsoever)
- So $x(t)$ could be the path of variable x in either case...but pay attention some equations change when using continuous instead of discrete time.
- Here we will use x_t for discrete time and $x(t)$ for continuous (do the same in your notes, exams, etc.)

Relating prices and interest rates

- P_t is the price of the final good at time t, normalize the price of the final good to 1 *in all periods*.
- Building on an insight by Kenneth Arrow (Arrow, 1964) that it is sufficient to price *securities* (assets) that transfer one unit of consumption from one date (or state of the world) to another.
- In Arrow-Debreu [A-D] economy prices P_t are linked to P_{t+1} by the *real rate of interest* from t to $t+1$ of r_{t+1}:

$$P_t / P_{t+1} = 1 + r_{t+1}$$

- If $P_t = 1$, then r_t is intertemporal exchange rate; "real" or "commodity" rate of interest $\equiv$ interest rate implied by prices.
- r_t will enable us to normalize the price to 1 in every period.

Market Structure, Endowments and Market Clearing V

A-D General equilibrium economies:

- The same good at different dates is a different commodity.
- Therefore, there will be *an infinite number of commodities*.
- Assume capital, K, depreciates by constant rate, "exponential form," at $\delta \in (0, 1)$: of 1 unit of capital at t, only $1 - \delta$ is left at $t + 1$.
- δ affects household r_t (rate of return for savings); indifference between lending and investing implies: $1 + r_t = R_t + (1 - \delta)$.
- *Interest rate* faced by the household will be $r_t = R_t - \delta$.

Firm Optimization I

- Consider the problem of profit maximization at a *representative firm*:

$$\pi_t \equiv \max_{L_t \geq 0, K_t \geq 0} F[K_t, L_t, A_t] - w_t L_t - R_t K_t. \tag{5}$$

- Since there are no irreversible investments or costs of adjustments, the production side can be represented as a static maximization problem.
- Equivalently, *cost minimization problem.*

Firm Optimization II

- Features worth noting:
  1. Problem is set up in terms of aggregate variables.
  2. Nothing multiplying the F term, price of the final good has normalized to 1.
  3. Already imposes competitive factor markets: firm is taking as given w_t and R_t.
  4. Concave problem, since F is concave.
- Since $F(\cdot)$ satisfies CRS, then either $\pi = 0$ or the solution does not exist.

Firm Optimization III

- In particular, given w_t, R_t and A_t the Firms demand for L_t and K_t is given by the solution to the FOC.
- Since F is differentiable, first-order necessary conditions imply:

$$w_t = F_L[K_t, L_t, A_t] > 0, \tag{6}$$

and

$$R_t = F_K[K_t, L_t, A_t] > 0. \tag{7}$$

- Note also that in (6) and (7), we used K_t and L_t, the amount of capital and labor used by firms.
- In fact, solving for K_t and L_t, we can derive the capital and labor demands of firms in this economy at rental prices R_t and w_t.
- Thus we could have used K_t^d instead of K_t, but this additional notation is not necessary.

Firm Optimization IV

- Alternative solution uses Assumption 1, $F(\cdot)$ is homogeneous of degree 1 in L_t and K_t, and Euler's Theorem with $m = 1$ is CRS:

$$\begin{aligned} Y_t &= F\left[K_t, L_t, A_t\right] \\ &= F_L\left[K_t, L_t, A_t\right] \cdot L_t + F_K\left[K_t, L_t, A_t\right] \cdot K_t \end{aligned}$$

$$\begin{aligned} \pi_t &= F_L\left(\cdot\right) \cdot L_t + F_K\left(\cdot\right) \cdot K_t - w_t L_t - R_t K_t \\ &= \left\{F_L\left(\cdot\right) - w_t\right\} \cdot L_t + \left\{F_K\left(\cdot\right) - R_t\right\} \cdot K_t \end{aligned}$$

- Where in equilibrium must have $\pi_t = 0$, which will only hold if both (6) and (7) hold, and $L^* = L_t = \bar{L}_t$ and $K^* = K_t = \bar{K}_t$.

Firm Optimization V

Proposition (1) Suppose Assumption 1 holds. Then in an equilibrium of a Solow growth model, firms make no profits, and in particular,

$$Y_t = w_t L_t + R_t K_t.$$

- **Proof:** As above Euler Thm; substitute (6) and (7) above into Y_t.
- Thus $\pi_t = 0$, so we do not need to specify firm ownership.

Fundamental Law of Motion of the Solow Model I

- Recall that K depreciates exponentially at the rate δ, so

$$K_{t+1} = (1-\delta) K_t + I_t \;\Leftrightarrow\; \triangle K_{t+1} = I_t - \delta K_t, \tag{8}$$

  where I_t is investment at time t.

- From national income accounting for a closed economy,

$$Y_t = C_t + I_t, \tag{9}$$

- Using (1), (8) and (9), any *feasible* dynamic allocation in this economy must satisfy

$$K_{t+1} \leq F[K_t, L_t, A_t] + (1-\delta) K_t - C_t \;\; \forall t \in \mathbb{N} \textit{ or } \mathbb{Z}_+$$

Fundamental Law of Motion of the Solow Model II

- Solow model is a mixture of an old-style Keynesian model and a modern dynamic macroeconomic model.
- Households do not optimize, but firms still maximize and factor markets clear.
- Note this is not derived from the maximization of utility function: welfare comparisons have to be taken with a grain of salt.
- *Behavioral rule* of the constant saving rate simplifies the structure of equilibrium considerably.

Fundamental Law of Motion of the Solow Model III

- Since the economy is closed (and there is no government spending),

$$S_t = I_t = Y_t - C_t.$$

- Individuals are assumed to save a constant fraction s of their income,

$$S_t = sY_t, \tag{10}$$

$$C_t = (1 - s)\,Y_t \tag{11}$$

- Implies that the supply of capital resulting from households' behavior can be expressed as

$$K_t^s = (1 - \delta)K_t + S_t = (1 - \delta)K_t + sY_t.$$

Fundamental Law of Motion of the Solow Model IV

- Setting supply and demand equal to each other, this implies $K_t^s = K_t$.
- From (3), we have $L_t = \bar{L}_t$.
- Combining these market clearing conditions with (1) and (8), we obtain *the fundamental law of motion* the Solow growth model:

$$K_{t+1} = sF\left[K_t, L_t, A_t\right] + (1-\delta) K_t. \tag{12}$$

- Nonlinear *difference equation*.
- Equilibrium of the Solow growth model is described by this equation together with laws of motion for L_t (or $\bar{L}_t$) and A_t.

Definition of Equilibrium I

Definition (2) In the basic Solow model for a given sequence of $\{L_t, A_t\}_{t=0}^{\infty}$ and an initial capital stock K_0, an equilibrium path is a sequence of capital stocks, output levels, consumption levels, wages and rental rates $\{K_t, Y_t, C_t, w_t, R_t\}_{t=0}^{\infty}$ such that K_t satisfies (12), Y_t is given by (1), w_t and R_t are given by (6) and (7), and C_t is given by (11).

- $K_{t+1} = sF[K_t, L_t, A_t] + (1-\delta) K_t$, $\quad Y_t = F[K_t, L_t, A_t]$,
  $w_t = F_L[K_t, L_t, A_t] > 0$, $\quad R_t = F_K[K_t, L_t, A_t] > 0$,
  $C_t = (1-s) Y_t$, $\quad I_t = S_t = sY_t$, $\quad r_t = R_t - \delta$
- Note an equilibrium is defined as an entire path of allocations and prices: *not* a static object.

Equilibrium Without Population Growth and Technological Progress I

- Make some further assumptions, which will be relaxed later:
  1. There is no population growth; total population is constant at some level $L > 0$. Since individuals supply labor inelastically, $L_t = L$.
  2. No technological progress, so that $A_t = A$.
- Define the capital-labor ratio of the economy as:

$$k_t \equiv \frac{K_t}{L}, \tag{13}$$

- Using $A_t = A$, $L_t = L$, and the constant returns to scale assumption [CRS], we can express output (income) per capita, $y_t \equiv Y_t/L$, as:

$$\begin{aligned} y_t &= \frac{1}{L}F\left(K_t, L, A\right) = \frac{L}{L}F\left(\frac{K_t}{L}, 1, A\right) = F\left(\frac{K_t}{L}, 1, A\right) \\ &\equiv f\left(k_t\right) \end{aligned} \tag{14}$$

Equilibrium Without Population Growth nor Technological Progress II

$$c_t = \frac{C_t}{L} = (1-s)\frac{Y_t}{L} = (1-s)\, y_t$$

- Note that $f(k_t)$ here depends on A, so we could write $f(k_t, A)$; but A is constant and can be normalized to $A = 1$.
- From the Euler Theorem,

$$\begin{aligned} R_t &= F_K(K, L, A) = F_K\left(\frac{K}{L}, 1, A\right) = f'(k_t) > 0 \text{ and} \\ w_t &= F_L(K, L, A) = f(k_t) - k_t f'(k_t) > 0. \end{aligned} \tag{15}$$

- Both are positive from Assumption 1.

Example:The Cobb-Douglas Production Function I

- Cobb-Douglas is a very special production function and many interesting phenomena are ruled out, but it is widely used:

$$\begin{aligned} Y_t &= F\left[K_t, L_t, A_t\right] \\ &= AK_t^{\alpha}L_t^{1-\alpha} \quad , 0 < \alpha < 1. \end{aligned} \tag{16}$$

- Satisfies Assumptions 1 and 2.
- Dividing both sides by L_t,

$$y_t = F\left(k_t, 1, A\right) = Ak_t^{\alpha},$$

- From equation (15),

$$R_t = \frac{\partial Ak_t^{\alpha}}{\partial k_t} = \alpha A k_t^{-(1-\alpha)}.$$

- From the Euler Theorem,

$$w_t = y_t - R_t k_t = (1-\alpha)Ak_t^{\alpha}.$$

Example:The Cobb-Douglas Production Function II

- Alternatively, in terms of the original Cobb-Douglas production function (16),

$$\begin{aligned} R_t &= \alpha A K_t^{\alpha-1} L_t^{1-\alpha} \\ &= \alpha A k_t^{-(1-\alpha)}, \end{aligned}$$

- Similarly, from (15),

$$\begin{aligned} w_t &= A k_t^{\alpha} - k_t \cdot \alpha A k_t^{-(1-\alpha)} \\ &= (1-\alpha) A K_t^{\alpha} L_t^{-\alpha}, \\ &= (1-\alpha) A k_t^{\alpha}, \end{aligned}$$

- Which verifies the alternative expression for the wage rate in (6)

Equilibrium Without Population Growth and Technological Progress III

- The per capita representation of the aggregate production function enables us to divide both sides of (12) by L to obtain:

$$k_{t+1} = sf(k_t) + (1-\delta) k_t. \tag{17}$$

- Since it is derived from (12), it also can be referred to as the *equilibrium difference equation* of the Solow model
- The other equilibrium quantities can be obtained from the capital-labor ratio k_t.

Definition (3) A steady-state equilibrium without technological progress and population growth is an equilibrium path in which $k_t = k^*$ for all t.

- The economy will tend to this steady state equilibrium over time (but never reach it in finite time).

Steady-State Capital-Labor Ratio

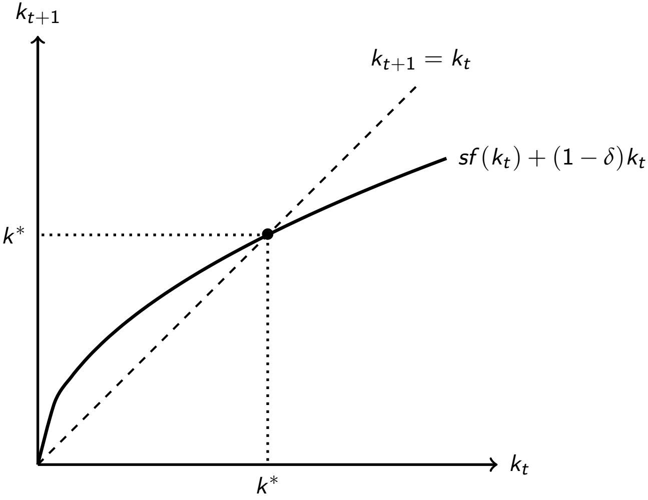


Figure 2.2 – Determination of the steady-state capital-labor ratio in the Solow model without population growth and technological change.

Equilibrium Without Population Growth and Technological Progress IV

- Curve represents (17) and the dashed line corresponds to the 45° line.
- Their (positive) intersection gives the steady-state value of the capital-labor ratio k^*,

$$\begin{aligned} k^* &= s \cdot f(k^*) + (1-\delta) k^* && \Longleftrightarrow \\ \delta k^* &= s \cdot f(k^*) && \Longleftrightarrow \\ \frac{f(k^*)}{k^*} &= \frac{\delta}{s}. && \qquad (18) \end{aligned}$$

- There is another intersection at $k = 0$, because the figure assumes that $f(0) = 0$. $k^* = 0$ is always a steady state [SS], but we will ignore this intersection throughout:
  1. If capital is not essential, $f(0)$ will be positive and $k = 0$ will cease to be a steady state equilibrium
  2. This intersection, even when it exists, is an *unstable point*
  3. It has no economic interest for us.

Unique Steady State: Basic Solow Model: $f(0) = \varepsilon > 0$.

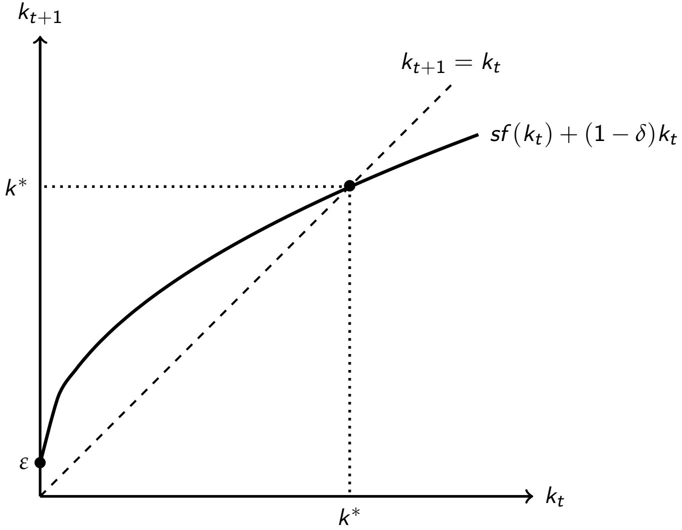


Figure 2.3 – Unique steady state in the basic Solow model when $f(0) = \varepsilon > 0$.

Equilibrium Without Population Growth and Technological Progress V

- Figure 2.4 is an alternative visual representation of the steady state: intersection between δk and the function $s \cdot f(k)$. Useful because:
  1. Depicts the levels of consumption and investment in a single figure.
  2. Emphasizes the steady-state equilibrium sets investment, $s \cdot f(k)$, equal to the amount of capital that needs to be "replenished", δk.
  3. Production: $f(k_t)$
  4. Consumption: $c_t = f(k_t) - s \cdot f(k_t) = (1-s) \cdot f(k_t)$
  5. Investment = savings: $i_t = s \cdot f(k_t)$
  6. Steady state capital k^* such that: $s \cdot f(k_t) = \delta k_t \equiv s \cdot f(k) = \delta k$.

Consumption and Investment in Steady State

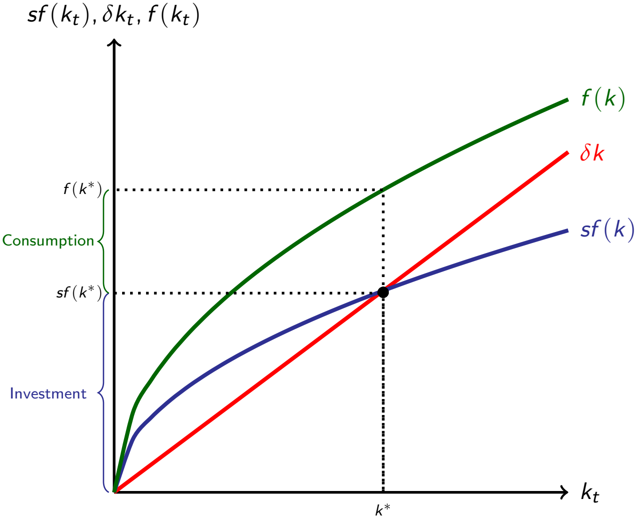


Figure 2.4 – Investment and consumption in the steady state equilibrium.

Equilibrium Without Population Growth and Technological Progress V

- Figure 2.4b is an alternate visualization, rate of change in capital, γ_k:
  1. Starting from (17): $k_{t+1} = s \cdot f(k_t) + (1-\delta) k_t$
  2. Rate of change of capital: $\gamma_{k_{t+1}} = \frac{k_{t+1}-k_t}{k_t} = \frac{\triangle k_t}{k_t} = \frac{s \cdot f(k_t)}{k_t} - \delta$
  3. In graph distance of $s \cdot f(k_t) / k_t$ from δ is rate of change.
- Supplemental Graphs

Rate of Change in Capital

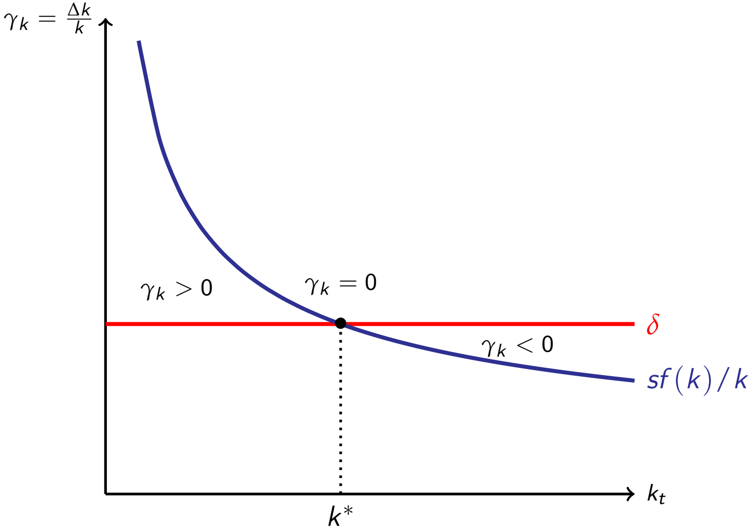


Figure 2.4B – Distance of $s \cdot f(k_t)/k_t$ from δ is rate of change of capital.

Rate of Change in Capital

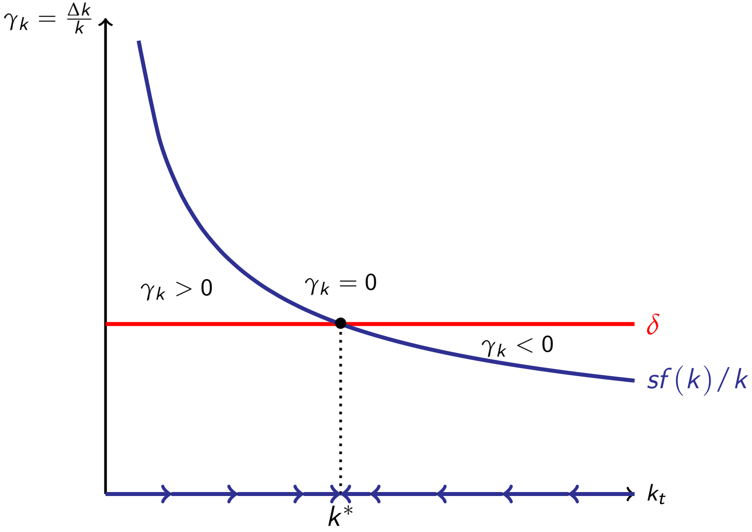


Figure 2.4B – Distance of $s \cdot f(k_t) / k_t$ from δ is rate of change of capital.

Equilibrium Without Population Growth and Technological Progress VI

Proposition (2) Consider the basic Solow growth model and suppose that Assumptions 1 and 2 hold. Then there exists a unique steady state equilibrium where: the capital-labor ratio $k^* \in (0, \infty)$ is given by:

$$\frac{f\left(k^*\right)}{k^*} = \frac{\delta}{s} \tag{18}$$

per capita output is given by

$$y^* = f\left(k^*\right) \tag{19}$$

and per capita consumption is given by

$$c^* = \left(1 - s\right) f\left(k^*\right). \tag{20}$$

Proof of Theorem

Existence:

- The preceding argument establishes that any k^* that satisfies (18) is a steady state.
- To establish existence, note that by Assumption 2 (and from L'Hospital's rule), $\lim_{k\to 0} f(k)/k = \infty$ and $\lim_{k\to\infty} f(k)/k = 0$.
- Moreover, $f(k)/k$ is continuous by Assumption 1, so by the Intermediate Value Theorem there exists k^* such that (18) is satisfied.

Proof of Theorem

Uniqueness:

- Differentiate $f(k)/k$ with respect to k, which gives

$$\frac{\partial [f(k)/k]}{\partial k} = \frac{f'(k)k - f(k)}{k^2} = -\frac{w}{k^2} < 0, \tag{21}$$

  where the last equality uses (15).
- Since $f(k)/k$ is everywhere (strictly) decreasing, there can only exist a unique value k^* that satisfies (18).
- Equations (19) and (20) then follow by definition.

Examples of nonexistence and nonuniqueness of interior steady states when Assumptions 1 and 2 are not satisfied.

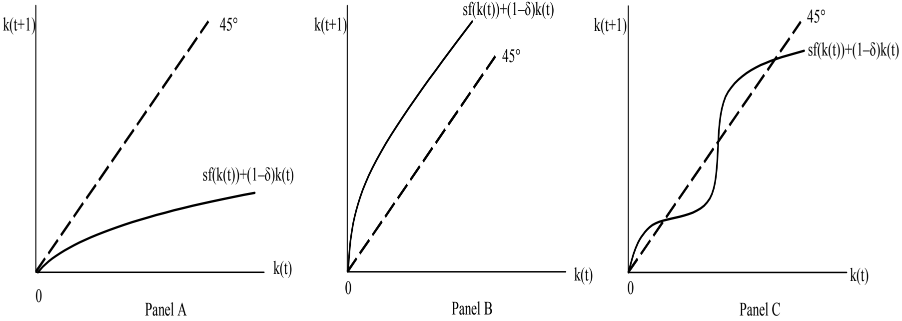


Figure 2.5 – Examples of nonexistence and nonuniqueness of interior steady states when Assumptions 1 and 2 are not satisfied.

Equilibrium Without Population Growth and Technological Progress VII

- Figure shows through a series of examples why Assumptions 1 and 2 cannot be dispensed with for the existence and uniqueness results.
- (A) and (B): the failure of Assumption 2 leads to a situation in which there is no steady state equilibrium with positive activity.
- (C): the failure of Assumption 1 leads to non-uniqueness of steady states.

Equilibrium Without Population Growth and Technological Progress VIII

- Generalize the production function in one simple way, and assume that

$$f\left(k\right) = A\tilde{f}\left(k\right),$$

- $A > 0$, so that A is a ("Hicks-neutral") shift parameter, with greater values corresponding to greater productivity of factors..
- Since $f\left(k\right)$ satisfies the regularity conditions imposed above, so does $\tilde{f}\left(k\right)$.

Equilibrium Without Population Growth and Technological Progress IX

- Comparative statics with respect to s, A and δ are straightforward for k^* and y^*.

Proposition (3) Suppose Assumptions 1 and 2 hold and $f(k) = A\tilde{f}(k)$. Denote the steady-state level of the capital-labor ratio by $k^*(A, s, \delta)$ and the steady-state level of output by $y^*(A, s, \delta)$ when the underlying parameters are A, s and δ. Then we have

$$\begin{aligned} \frac{\partial k^*(\cdot)}{\partial A} &> 0, \ \frac{\partial k^*(\cdot)}{\partial s} > 0 \text{ and } \frac{\partial k^*(\cdot)}{\partial \delta} < 0 \\ \frac{\partial y^*(\cdot)}{\partial A} &> 0, \ \frac{\partial y^*(\cdot)}{\partial s} > 0 \text{ and } \frac{\partial y^*(\cdot)}{\partial \delta} < 0. \end{aligned}$$

- Using Figure 2.4 by varying A, s and δ the effects on k^* and y^* can be demonstrated.

Varying A, s and δ: effects on k^* and y^*

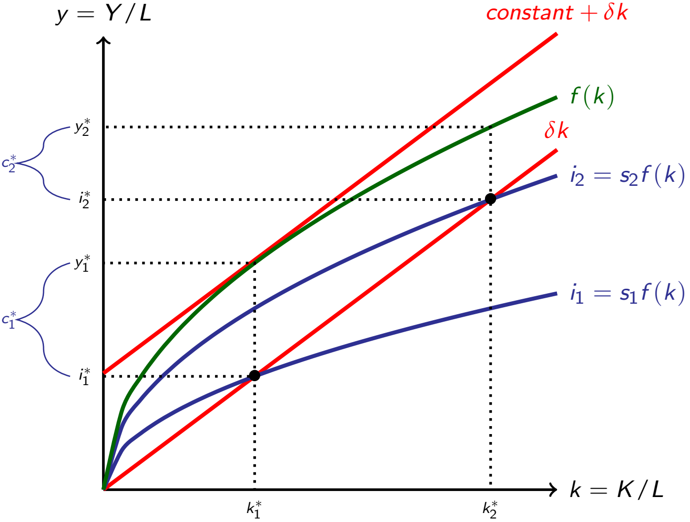


Figure 2.5 – Examples of varying s to determine the effects on k^* and y^*.

Equilibrium Without Population Growth and Technological Progress IX

- **Proof of comparative static results:** follows immediately by writing

$$\frac{\tilde{f}\left(k^{*}\right)}{k^{*}}=\frac{\delta}{As}$$

Now apply the implicit function theorem to obtain the results.

- For example,

$$\frac{\partial k^{*}}{\partial s}=\frac{\delta\left(k^{*}\right)^{2}}{s^{2}w^{*}}>0$$

where $w^{*}=f\left(k^{*}\right)-k^{*}f'\left(k^{*}\right)>0$.

- The other results follow similarly.

Equilibrium Without Population Growth and Technological Progress X

- Same comparative statics with respect to A and δ immediately apply to c^* as well. $\partial c^* / \partial \delta < 0$ and $\partial c^* / \partial A > 0$
- But c^* will not be monotone in the saving rate (think, for example, of $s = 1$). $\partial c^* / \partial s \overset{?}{\lesseqgtr} 0$
- In fact, there will exist a specific level of the saving rate, s_{gold}, referred to as the "golden rule" saving rate, which maximizes c^*.
- But cannot always say whether the golden rule saving rate is "better" than some other saving rate.
- Write the steady state relationship between c^* and s and suppress the other parameters:

$$\begin{aligned} c^*(s) &= (1-s) f(k^*(s)), \\ &= f(k^*(s)) - \delta k^*(s), \end{aligned}$$

- The second equality exploits that in steady state $s \cdot f(k^*) = \delta k^*$.

Equilibrium Without Population Growth and Technological Progress XI

- Differentiating with respect to s,

$$\frac{\partial c^*(s)}{\partial s} = \left[f'(k^*(s)) - \delta\right] \frac{\partial k^*}{\partial s}. \tag{22}$$

- s_{gold} is such that $\partial c^*(s_{gold}) / \partial s = 0$ (FOC) (Verify SOC holds). The corresponding steady-state golden rule capital stock is defined as k^*_{gold}.

Proposition (4) In the basic Solow growth model, the highest level of steady-state consumption is reached for s_{gold}, with the corresponding steady state capital level k^*_{gold} such that

$$f'\left(k^*_{gold}\right) = \delta. \tag{23}$$

The “Golden Rule”

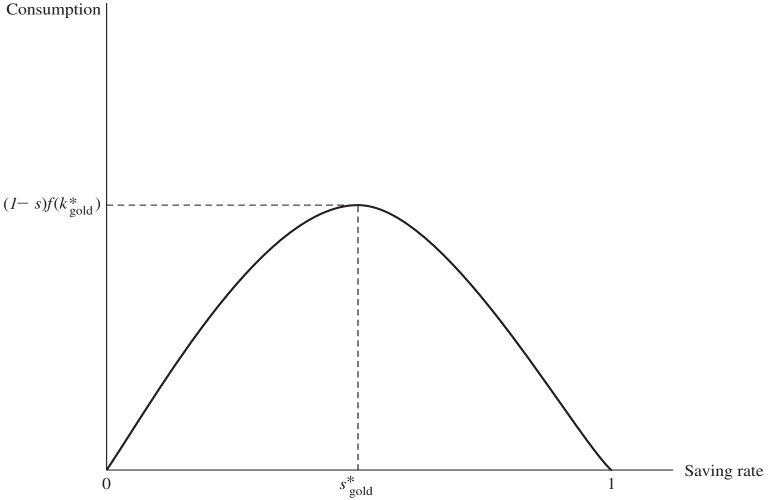
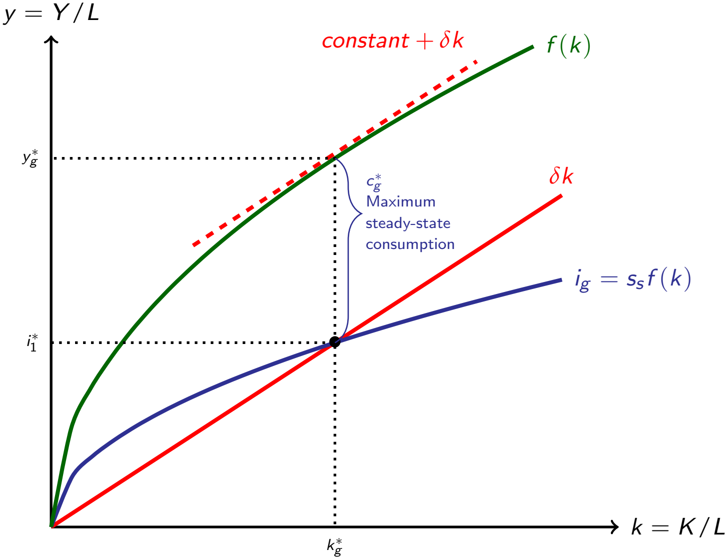


Figure 2.6 – The “golden rule” level of savings rate, which maximizes steady-state consumption.

The "Golden Rule"- Graphical Solution



Proof of Proposition: Golden Rule

- By definition $\partial c^* (s_{gold}) / \partial s = 0$.
- From Proposition above, $\partial k^* / \partial s > 0$, thus (22) can be equal to zero only when $f' (k^* (s_{gold})) = \delta$.
- Moreover, when $f' (k^* (s_{gold})) = \delta$, it can be verified that $\partial^2 c^* (s_{gold}) / \partial s^2 < 0$, so $f' (k^* (s_{gold})) = \delta$ indeed corresponds a local maximum.
- That $f' (k^* (s_{gold})) = \delta$ also yields the global maximum is a consequence of the following observations:
  - $\forall\ s \in [0, 1]$ we have $\partial k^* / \partial s > 0$ and moreover, when $s < s_{gold}$, $f' (k^* (s)) - \delta > 0$ by the concavity of f, so $\partial c^* (s) / \partial s > 0$ for all $s < s_{gold}$.
  - by the converse argument, $\partial c^* (s) / \partial s < 0$ for all $s > s_{gold}$.
  - Therefore, only s_{gold} satisfies $f' (k^* (s)) = \delta$ and gives the unique global maximum of SS consumption per capita.

Equilibrium Without ... XII: Dynamic Inefficiency

- When the economy is below k^*_{gold}, higher saving will increase SS consumption; when it is above k^*_{gold}, steady-state SS consumption can be increased by saving less.
- When economy is above k^*_{gold}, capital-labor ratio is too high so that individuals are investing too much and not consuming enough. This problem is called *dynamic inefficiency*, (clearly not Pareto Optimal) since we can increase consumption in all periods!
- When economy is below k^*_{gold}, although a higher steady-state consumption can be reached, the path involves a period of higher savings and lower consumption, not clear if dynamically inefficient.
- Still...without intertemporal utility function to measure, statements about "inefficiency" have to be considered with caution.
- Such dynamic inefficiency will not arise in the Neoclassical Growth Model once we endogenize consumption-saving decisions.

The "Golden Rule"- *Dynamic Inefficiency*

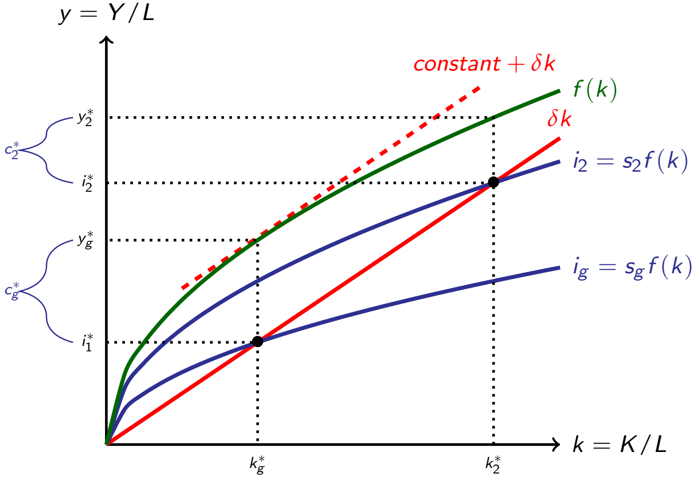


Figure 2.6B – In case economy is above k^*_{gold}, capital-labor ratio is too high so that individuals are investing too much and not consuming enough (*dynamic inefficiency)*, as all periods are increased by consuming more.

Section 3

Transitional Dynamics in the Discrete Time Solow Model

Review of the Discrete-Time Solow Model

- Per capita capital stock evolves according to (17):

$$k_{t+1} = sf(k_t) + (1-\delta)k_t.$$

- The steady-state value of the capital-labor ratio k^* is given by (18):

$$\frac{f(k^*)}{k^*} = \frac{\delta}{s}.$$

- Consumption is given by (20):

$$c_t = (1-s)y_t$$

- And factor prices are given by (15):

$$\begin{aligned} R_t &= f'(k_t) > 0 \text{ and} \\ w_t &= f(k_t) - k_t f'(k_t) > 0. \end{aligned}$$

Transitional Dynamics

- *Equilibrium path*: not simply steady state, but entire path of capital stock, output, consumption and factor prices.
  - In engineering and physical sciences, equilibrium is point of rest of dynamical system, thus *the steady state equilibrium*.
  - In economics, non-steady-state behavior also governed by optimizing behavior of households and firms and market clearing.
- Need to study the "transitional dynamics" of the equilibrium difference equation (17) starting from an arbitrary initial capital-labor ratio $k(0) > 0$.
- Key question: whether economy will tend to steady state and how it will behave along the transition path.

Transitional Dynamics: Review I

- Consider the nonlinear system of autonomous difference equations,

$$\mathbf{x}_{t+1} = \mathbf{G}(\mathbf{x}_t), \tag{24}$$

- $\mathbf{x}_t \in \mathbb{R}^n$ and $\mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n$.
- Let $\mathbf{x}^*$ be a fixed point of the mapping $\mathbf{G}(\cdot)$, i.e.,

$$\mathbf{x}^* = \mathbf{G}(\mathbf{x}^*).$$

- $\mathbf{x}^*$ is sometimes referred to as "an equilibrium point" of (24).
- We will refer to $\mathbf{x}^*$ as a stationary point or a *steady state* of (24).

Definition (4) A steady state $\mathbf{x}^*$ is (locally) *asymptotically stable* if there exists an open set $B(\mathbf{x}^*) \ni \mathbf{x}^*$ such that for any solution $\{\mathbf{x}_t\}_{t=0}^{\infty}$ to (24) with $\mathbf{x}(0) \in B(\mathbf{x}^*)$, we have $\mathbf{x}_t \to \mathbf{x}^*$. Moreover, $\mathbf{x}^*$ is *globally asymptotically stable* if for all $\mathbf{x}(0) \in \mathbb{R}^n$, for any solution $\{\mathbf{x}_t\}_{t=0}^{\infty}$, we have $\mathbf{x}_t \to \mathbf{x}^*$.

Transitional Dynamics: Review II

Simple Result About Stability

- Let $x_t, a, b \in \mathbb{R}$, then the unique steady state of the linear difference equation $x_{t+1} = ax_t + b$ is globally asymptotically stable (in the sense that $x_t \to x^* = b/\left(1-a\right)$) if $|a| < 1$.

- Suppose that $g : \mathbb{R} \to \mathbb{R}$ is differentiable at the steady state x^*, defined by $g\left(x^*\right) = x^*$. Then, the steady state of the nonlinear difference equation $x_{t+1} = g\left(x_t\right)$, x^*, is locally asymptotically stable if $\left|g'\left(x^*\right)\right| < 1$. Moreover, if $\left|g'\left(x\right)\right| < 1$ for all $x \in \mathbb{R}$, then x^* is globally asymptotically stable.

Transitional Dynamics in the Discrete Time Solow Model I

Now we can analyze the stability of the Solow growth model difference equation (17): $k_{t+1} = sf(k_t) + (1-\delta) k_t$.

Proposition (5) Suppose that Assumptions 1 and 2 hold, then the steady-state equilibrium of the Solow growth model described by the difference equation (17) is globally asymptotically stable, and starting from any $k(0) > 0$, k_t monotonically converges to k^*.

Proof of Proposition: Transitional Dynamics I

- Let $g(k) \equiv sf(k) + (1-\delta)k$. First observe that $g'(k)$ exists and is always strictly positive, i.e., $g'(k) > 0$ for all k.
- Next, from (17) [$k_{t+1} = sf(k_t) + (1-\delta)k_t$],

$$k_{t+1} = g(k_t), \tag{27}$$

  with a unique steady state at k^*.
- From (18), the steady-state capital k^* satisfies $\delta k^* = s \cdot f(k^*)$, or

$$k^* = g(k^*). \tag{28}$$

- Recall that $f(\cdot)$ is concave and differentiable from Assumption 1 and satisfies $f(0) \geq 0$ from Assumption 2.

Proof of Proposition: Transitional Dynamics II

- For any strictly concave differentiable function,

$$f(k) > f(0) + kf'(k) \geq kf'(k), \tag{29}$$

- The second inequality uses the fact that $f(0) \geq 0$.
- Since (29) implies that $\delta = sf(k^*)/k^* > sf'(k^*)$, we have $g'(k^*) = sf'(k^*) + 1 - \delta < 1$. Therefore,

$$g'(k^*) \in (0, 1).$$

- The Simple Result then establishes local asymptotic stability.

Proof of Proposition: Transitional Dynamics III

- To prove global stability, note that for all $k_t \in (0, k^*)$,

$$\begin{aligned} k_{t+1} - k^* &= g(k_t) - g(k^*) \\ &= -\int_{k_t}^{k^*} g'(k)\, dk, \\ &< 0 \end{aligned}$$

- First line follows by subtracting (28) from (27), second line uses the fundamental theorem of calculus, and third line follows from the observation that $g'(k) > 0$ for all k.

Proof of Proposition: Transitional Dynamics IV

- Next, (17) also implies

$$\begin{aligned}
\frac{k_{t+1} - k_t}{k_t} &= s\frac{f(k_t)}{k_t} - \delta \\
&> s\frac{f(k^*)}{k^*} - \delta \\
&= 0,
\end{aligned}$$

- Second line uses the fact that $f(k)/k$ is decreasing in k (from (29) above) and last line uses the definition of k^*.
- These two arguments together establish that for all $k_t \in (0, k^*)$, $k_{t+1} \in (k_t, k^*)$.
- An identical argument implies that for all $k_t > k^*$, $k_{t+1} \in (k^*, k_t)$.
- Therefore, $\{k_t\}_{t=0}^{\infty}$ monotonically converges to k^* and is globally stable.

Transitional Dynamics in the Discrete Time Solow Model II

- Stability result can be seen diagrammatically in the Figure:
  - Starting from initial capital stock $k_0 < k^*$, economy grows towards k^*, *capital deepening* and growth of per capita income.
  - If economy were to start with $k_0' > k^*$, reach the steady state by decumulating capital and contracting.

Proposition (6) Suppose that Assumptions 1 and 2 hold, and $k_0 < k^*$, then $\{w_t\}_{t=0}^{\infty}$ is an increasing sequence and $\{R_t\}_{t=0}^{\infty}$ is a decreasing sequence. If $k_0 > k^*$, the opposite results apply.

- Thus far Solow growth model has a number of nice properties, but no growth, except when the economy starts with $k_0 < k^*$.

Transitional Dynamics in Figure

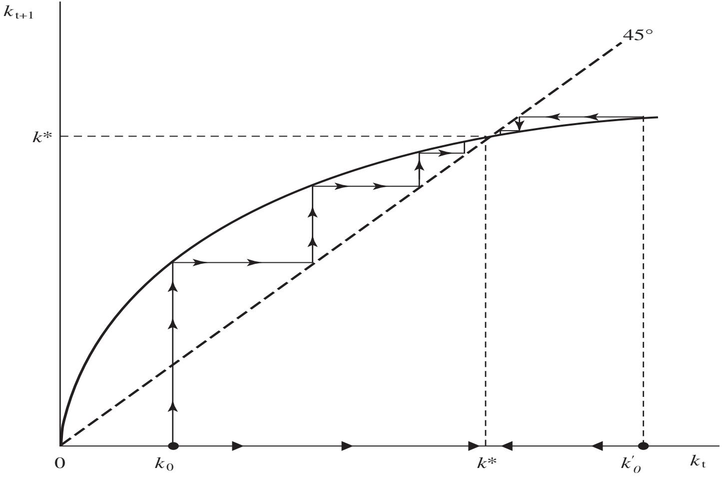


Figure 2.7 – Transitional dynamics in the basic Solow model.

Section 4

The Solow Model in Continuous Time

From Difference to Differential Equations I

- Start with a simple difference equation

$$x_{t+1} - x_t = g\left(x_t\right). \tag{30}$$

- Now consider the following approximation for any $\Delta t \in \left[0, 1\right]$,

$$x_{t+\Delta t} - x_t \simeq \Delta t \cdot g\left(x_t\right),$$

- When $\Delta t = 0$, this equation is just an identity. When $\Delta t = 1$, it gives (30).
- In-between it is a linear approximation, not too bad if $g\left(x\right) \simeq g\left(x_t\right)$ for all $x \in \left[x_t, x_{t+1}\right]$

From Difference to Differential Equations II

- Divide both sides of this equation by Δt, and take limits

$$\lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \dot{x}(t) \simeq g(x(t)), \tag{31}$$

where

$$\dot{x}(t) \equiv \frac{dx(t)}{dt}$$

- Equation (31) is a differential equation representing (30) for the case in which t and $t+1$ is "small".

Fundamental Eq. of Solow Model in Continuous Time I

- Nothing has changed on the production side, so (15) still give the factor prices, now interpreted as instantaneous wage and rental rates. $R(t) = f'(k(t)) > 0$ and $w(t) = f(k(t)) - k(t) f'(k(t)) > 0$.
- Savings are again: $S(t) = sY(t)$,
- Consumption is given by (11) above: $C(t) = (1-s)Y(t)$

Fundamental Eq. of Solow Model in Continuous Time II

- Introduce population growth (constant fertility rate [cfr]),
$$L(t) = e^{nt} L(0). \tag{32}$$
- This directly leads to the growth rate of population:
$$\dot{L}(t) / L(t) = n > 0$$
as
$$\dot{L}(t) = dL(t) / dt = n \cdot e^{nt} L(0) = n \cdot L(t)$$

Fundamental Eq. of Solow Model in Continuous Time III

- Recall

$$k(t) \equiv \frac{K(t)}{L(t)},$$

Implies

$$\begin{aligned}\frac{\dot{k}(t)}{k(t)} &= \frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)}, \\ &= \frac{\dot{K}(t)}{K(t)} - n.\end{aligned}$$

Fundamental Eq. of Solow Model in Continuous Time IV

- From the limiting argument leading to equation (31),

$$\dot{K}(t) = sF[K(t), L(t), A(t)] - \delta K(t).$$

- Using the definition of $k(t)$ and the constant returns to scale properties of the production function,

$$\frac{\dot{k}(t)}{k(t)} = s\frac{F[K, L, A]}{K(t)} - (n+\delta) = s\frac{f(k(t))}{k(t)} - (n+\delta), \qquad (33)$$

Fundamental Eq. of Solow Model in Continuous Time V

Definition (5) In the basic Solow model in continuous time with population growth at the rate n, no technological progress and an initial capital stock $K(0)$, an equilibrium path is a sequence of capital stocks, labor, output levels, consumption levels, wages and rental rates $[K(t), L(t), Y(t), C(t), w(t), R(t)]_{t=0}^{\infty}$ such that $L(t)$ satisfies (32), $k(t) \equiv K(t)/L(t)$ satisfies (33), $Y(t)$ is given by the aggregate production function, $C(t)$ is given by (11), and $w(t)$ and $R(t)$ are given by (15).

Fundamental Eq. of Solow Model in Continuous Time VI

- As before, in *steady-state* equilibrium $k(t)$ remains constant at k^*.
  - $L(t)$ satisfies (32):
  $$L(t) = e^{nt} L(0)$$
  - $k(t) \equiv K(t) / L(t)$ satisfies (33):
  $$\frac{\dot{k}(t)}{k(t)} = s\frac{f(k(t))}{k(t)} - (n + \delta)$$
  - Output per capita is given by
  $$y(t) = f(k(t))$$
  - $C(t)$ is given by (11):
  $$C(t) = (1 - s) Y(t) \quad \text{or} \quad c(t) = (1 - s) y(t)$$
  - and $w(t)$ and $R(t)$ are given by (15):
  $$R(t) = F_K = f'(k(t)) > 0 \quad \text{and} \quad w(t) = F_L = f(k(t)) - k(t) f'(k(t)) > 0.$$

Investment & consumption in S-S-M w/ population growth.

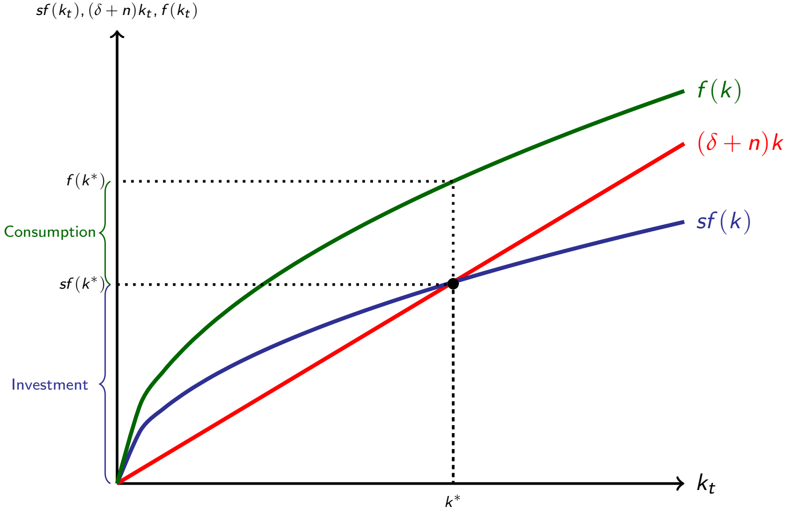


Figure 2.8 – Investment & consumption in the steady-state equilibrium with population growth.

Steady State of the Solow Model in Continuous Time I

- Equilibrium path (33) has a unique *steady state* at k^*, which is given by a slight modification of (18) above:

$$\frac{f(k^*)}{k^*} = \frac{n+\delta}{s}. \tag{34}$$

Proposition (7) Consider the basic Solow growth model in continuous time and suppose that Assumptions 1 and 2 hold. Then there exists a unique steady state equilibrium where the capital-labor ratio is equal to $k^* \in (0, \infty)$ and is given by (34), per capita output is given by

$$y^* = f(k^*)$$

and per capita consumption is given by

$$c^* = (1-s)\, f(k^*).$$

Steady State of the Solow Model in Continuous Time II

- Moreover, again defining $f(k) = A\tilde{f}(k)$, we obtain:

Proposition (8) Suppose Assumptions 1 and 2 hold and $f(k) = A\tilde{f}(k)$. Denote the steady-state equilibrium level of the capital-labor ratio by $k^*(A, s, \delta, n)$ and the steady-state level of output by $y^*(A, s, \delta, n)$ when the underlying parameters are given by A, s and δ. Then we have

$$
\begin{aligned}
\frac{\partial k^*(\cdot)}{\partial A} &> 0, \ \frac{\partial k^*(\cdot)}{\partial s} > 0, \ \frac{\partial k^*(\cdot)}{\partial \delta} < 0 \text{ and } \frac{\partial k^*(\cdot)}{\partial n} < 0 \\
\frac{\partial y^*(\cdot)}{\partial A} &> 0, \ \frac{\partial y^*(\cdot)}{\partial s} > 0, \ \frac{\partial y^*(\cdot)}{\partial \delta} < 0 \text{ and } \frac{\partial y^*(\cdot)}{\partial n} < 0.
\end{aligned}
$$

Steady State of the Solow Model in Continuous Time III

- Relative to the earlier $n = 0$ (Prop. 3) a higher population growth rate, $n > 0$, reduces the capital-labor ratio and output per capita.
- $n > 0 \rightarrow$ faster dilution of capital $\rightarrow$ lower SS capital-labor ratio.

Testable Implication:

countries with high population growth rates should be poorer.

Section 5

Transitional Dynamics in the Continuous Time Solow Model

Review Dynamical Systems I

- Analysis of transitional dynamics and stability with continuous time yields similar results to Theorems (2) and (3), with slightly simpler analysis.
- Recall basic results on stability of systems of differential equations.

Theorem (4) **(Stability of Linear Differential Equations)** Consider the following autonomous linear differential equation system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b} \tag{35}$$

with initial value $\mathbf{x}(0)$, where $\mathbf{x}(t) \in \mathbb{R}^n$ for all t, $\mathbf{A}$ is an invertible $n \times n$ matrix, and $\mathbf{b}$ is a $n \times 1$ column vector. Let $\mathbf{x}^*$ be the unique steady state of the system given by $\mathbf{A}\mathbf{x}^* + \mathbf{b} = 0$. Suppose that all eigenvalues of $\mathbf{A}$ have negative real parts. Then the steady state of the differential equation (35) $\mathbf{x}^*$ is globally asymptotically stable, in the sense that starting from any $\mathbf{x}(0) \in \mathbb{R}^n$, $\mathbf{x}(t) \to \mathbf{x}^*$.

Review Dynamical Systems II

Theorem (5) (Local Stability of Nonlinear Differential Equations) Consider the following nonlinear autonomous differential equation:

$$\dot{\mathbf{x}}(t) = \mathbf{G}\left(\mathbf{x}\left(t\right)\right) \tag{36}$$

with initial value $\mathbf{x}(0)$, where $\mathbf{G}: \mathbb{R}^n \to \mathbb{R}^n$. Let $\mathbf{x}^*$ be a steady state of this system, that is, $G(\mathbf{x}^*) = 0$, and suppose that $\mathbf{G}$ is differentiable at $\mathbf{x}^*$. Define

$$\mathbf{A} \equiv D\mathbf{G}(\mathbf{x}^*),$$

and suppose that all eigenvalues of $\mathbf{A}$ have negative real parts. Then the steady state of the differential equation (36), $\mathbf{x}^*$, is locally asymptotically stable, in the sense that there exists an open neighborhood of $\mathbf{x}^*$, $\mathbf{B}(\mathbf{x}^*) \subset \mathbb{R}^n$, such that starting from any $\mathbf{x}(0) \in B(\mathbf{x}^*)$, $\mathbf{x}(t) \to \mathbf{x}^*$.

Once again an immediate corollary is as follows:

Review Dynamical Systems III

Corollary (2)

1. Let $x(t) \in \mathbb{R}$. Then the steady state of the linear differential equation $\dot{x}(t) = ax(t) + b$ is asymptotically globally stable (in the sense that $x(t) \to -b/a$) if $a < 0$.
2. Let $g : \mathbb{R} \to \mathbb{R}$ be differentiable in the neighborhood of the steady state x^* defined by $g(x^*) = 0$ and suppose that $g'(x^*) < 0$. Then the steady state of the nonlinear differential equation $\dot{x}(t) = g(x(t))$, x^* , is locally asymptotically stable.

Transitional Dynamics Continuous Time Solow Model I

Simple Result about Stability In Continuous Time Model

- Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function and suppose that there exists a unique x^* such that $g(x^*) = 0$.
  - Moreover, suppose $g(x) < 0$ for all $x > x^*$ and $g(x) > 0$ for all $x < x^*$.
  - Then the steady state of the nonlinear differential equation $\dot{x}(t) = g(x(t))$, x^*, is globally asymptotically stable, i.e., starting with any $x(0)$, $x(t) \to x^*$.

Note that $g(x)$ could be non-monotonic and still satisfy the first condition.

Dynamics of capital-labor ratio in basic Solow model

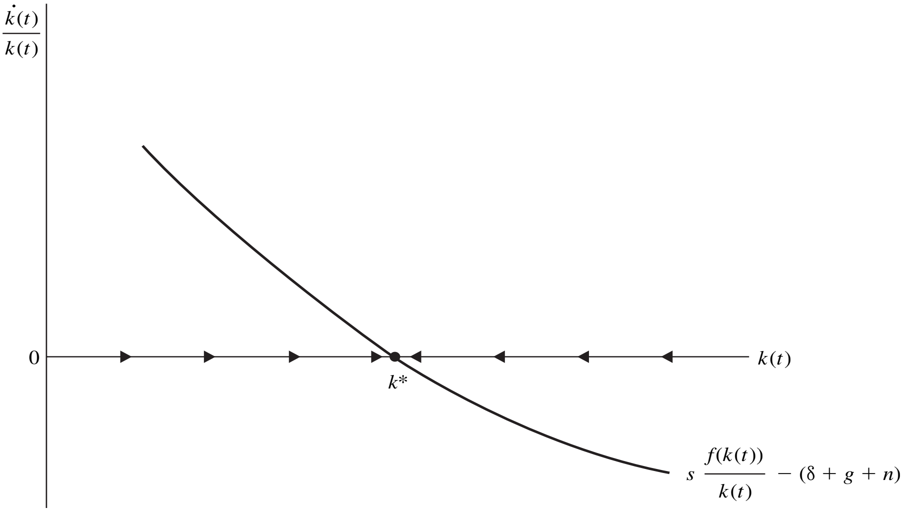


Figure 2.9 – Dynamics of the capital-labor ratio in the basic Solow model. – Simple Result.

Transitional Dynamics Continuous Time Solow Model II

Fundamental equation is

$$\dot{k}\left(t\right)=s\cdot f\left(k\left(t\right)\right)-\left(n+\delta\right)k\left(t\right),$$

so that

$$\frac{\partial\dot{k}\left(t\right)}{\partial k\left(t\right)}=s\cdot f'\left(k\left(t\right)\right)-\left(n+\delta\right).$$

Thus, SS is determined by

$$s\cdot f\left(k^{*}\right)=\left(n+\delta\right)k^{*}\ \rightarrow\ \left(n+\delta\right)=s\cdot f\left(k^{*}\right)/k^{*}$$

taking derivative wrt k^{*}

$$\frac{\partial\dot{k}}{\partial k}\left(k^{*}\right)=s\cdot f'\left(k^{*}\right)-\left(n+\delta\right)=s\cdot f'\left(k^{*}\right)-s\cdot f\left(k^{*}\right)/k^{*}$$

Transitional Dynamics Continuous Time Solow Model III

Using the SS condition, we have that

$$\frac{\partial \dot{k}}{\partial k}\left(k^*\right) = s \cdot \left[\frac{f'\left(k^*\right)k^* - f\left(k^*\right)}{k^*}\right] = -\frac{s \cdot w^*}{k^*} < 0$$

i.e.

$$\frac{\partial \dot{k}\left(k^*\right)}{\partial k} < 0.$$

Transitional Dynamics Continuous Time Solow Model VII

Proposition (9) Suppose that Assumptions 1 and 2 hold, then the basic Solow growth model in continuous time with constant population growth and no technological change is globally asymptotically stable, and starting from any $k(0) > 0$, $k(t) \rightarrow k^*$.

- **Proof:** Follows immediately from the Theorem and Corollary (2)(3.) above by noting whenever $k < k^*$, $sf(k) - (n+\delta)k > 0$ and whenever $k > k^*$, $sf(k) - (n+\delta)k < 0$.
- Figure 2.9: plots the right-hand side of (33) and makes it clear that whenever $k < k^*$, $\dot{k} > 0$ and whenever $k > k^*$, $\dot{k} < 0$, so k monotonically converges to k^*.

Dynamics with Cobb-Douglas Production Function I

- Return to the Cobb-Douglas Example

$$F[K, L, A] = AK^{\alpha}L^{1-\alpha} \text{ with } 0 < \alpha < 1.$$

- Special, mainly because elasticity of substitution between capital and labor is 1.
- Recall for a homothetic production function $F(K, L)$, the elasticity of substitution is

$$\sigma \equiv -\left[\frac{\partial \ln (F_K / F_L)}{\partial \ln (K / L)}\right]^{-1}, \tag{37}$$

- F is required to be homothetic, so that F_K / F_L is only a function of K / L.
- For the Cobb-Douglas production function $F_K / F_L = (\alpha / (1-\alpha)) \cdot (L / K)$, thus $\sigma = 1$.

Dynamics with Cobb-Douglas Production Function II

- When the production function is Cobb-Douglas and factor markets are competitive, equilibrium factor shares will be constant:

$$\begin{aligned}
\alpha_K(t) &= \frac{R(t)\,K(t)}{Y(t)} \\
&= \frac{F_K(K(t), L(t))\,K(t)}{Y(t)} \\
&= \frac{\alpha A\,[K(t)]^{\alpha-1}\,[L(t)]^{1-\alpha}\,K(t)}{A\,[K(t)]^{\alpha}\,[L(t)]^{1-\alpha}} \\
&= \alpha.
\end{aligned}$$

- Similarly, the share of labor is $\alpha_L(t) = 1 - \alpha$.
- Reason: with $\sigma = 1$, as capital increases, its marginal product decreases proportionally, leaving the capital share constant.

Dynamics with Cobb-Douglas Production Function III

- Per capita production function takes the form $f(k) = Ak^{\alpha}$, so the steady state is given again as

$$A(k^*)^{\alpha-1} = \frac{n+\delta}{s}$$

or

$$k^* = \left(\frac{sA}{n+\delta}\right)^{\frac{1}{1-\alpha}},$$

- k^* is increasing in s and A and decreasing in n and δ.
- In addition, k^* is increasing in α: higher α implies higher share paid to capital and less diminishing MPK
- Transitional dynamics are also straightforward in this case:

$$\dot{k}(t) = sA[k(t)]^{\alpha} - (n+\delta)k(t)$$

with initial condition $k(0)$.

Dynamics with Cobb-Douglas Production Function IV

- To solve this equation, let $x(t) \equiv k(t)^{1-\alpha}$,

$$\dot{x}(t) = (1-\alpha) sA - (1-\alpha)(n+\delta) x(t),$$

- General solution

$$x(t) = \frac{sA}{n+\delta} + \left[x(0) - \frac{sA}{n+\delta}\right] \exp(-(1-\alpha)(n+\delta)t).$$

- In terms of the capital-labor ratio

$$k(t) = \left\{\frac{sA}{n+\delta} + \left[[k(0)]^{1-\alpha} - \frac{sA}{n+\delta}\right] \exp(-(1-\alpha)(n+\delta)t)\right\}^{\frac{1}{1-\alpha}}.$$

Dynamics with Cobb-Douglas Production Function V

- This solution illustrates:
  - starting from any $k(0)$, $k(t) \to k^* = (sA/(n+\delta))^{1/(1-\alpha)}$, and rate of adjustment is related to $(1-\alpha)(n+\delta)$,
  - more specifically, gap between $k(0)$ and its steady-state value is closed at the exponential rate $(1-\alpha)(n+\delta)$.
- Intuitive:
  - higher α, less diminishing returns, slows down rate at which marginal and average product of capital declines, reduces rate of adjustment to steady state.
  - smaller δ and smaller n: slow down the adjustment of capital per worker and thus the rate of transitional dynamics.

Constant Elasticity of Substitution Production Function I

Constant Elasticity of Substitution – CES

- Imposes a constant elasticity, σ, not necessarily equal to 1.
- Consider a vector-valued index of technology $\mathbf{A}(t) = (A_H(t), A_K(t), A_L(t))$.
- CES production function can be written as

$$
\begin{aligned}
Y(t) &= F[K(t), L(t), \mathbf{A}(t)] \\
&\equiv A_H(t)\left[\gamma (A_K(t) K(t))^{\frac{\sigma-1}{\sigma}} + (1-\gamma)(A_L(t) L(t))^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \qquad (38)
\end{aligned}
$$

- $A_H(t) > 0$, $A_K(t) > 0$ and $A_L(t) > 0$ are three different types of technological change
- $\gamma \in (0, 1)$ is a distribution parameter,

Constant Elasticity of Substitution Production Function II

- $\sigma \in [0, \infty]$ is the elasticity of substitution: easy to verify that

$$\frac{F_K}{F_L} = \frac{\gamma A_K(t)^{\frac{\sigma-1}{\sigma}} K(t)^{-\frac{1}{\sigma}}}{(1-\gamma) A_L(t)^{\frac{\sigma-1}{\sigma}} L(t)^{-\frac{1}{\sigma}}},$$

- Thus, indeed have

$$\sigma = -\left[\frac{\partial \ln(F_K/F_L)}{\partial \ln(K/L)}\right]^{-1}.$$

Constant Elasticity of Substitution Production Function III

- As $\sigma \to 1$, the CES production function converges to the Cobb-Douglas

$$Y(t) = A_H(t)(A_K(t))^{\gamma}(A_L(t))^{1-\gamma}(K(t))^{\gamma}(L(t))^{1-\gamma}$$

- As $\sigma \to \infty$, the CES production function becomes linear, i.e.

$$Y(t) = \gamma A_H(t) A_K(t) K(t) + (1-\gamma) A_H(t) A_L(t) L(t).$$

- Finally, as $\sigma \to 0$, the CES production function converges to the Leontief production function with no substitution between factors,

$$Y(t) = A_H(t) \min\{\gamma A_K(t) K(t); (1-\gamma) A_L(t) L(t)\}.$$

- Leontief production function: if $\gamma A_K(t) K(t) \neq (1-\gamma) A_L(t) L(t)$, either capital or labor will be partially "idle".

Section 6

A First Look at Sustained Growth

A First Look at Sustained Growth I

- Cobb-Douglas already showed that when α is close to 1, adjustment to steady-state level can be very slow.
- Simplest model of sustained growth essentially takes $\alpha = 1$ in terms of the Cobb-Douglas production function above.
- Relax Assumptions 1 and 2 and suppose

$$F\left[K\left(t\right), L\left(t\right), A\left(t\right)\right] = AK\left(t\right), \tag{39}$$

  where $A > 0$ is a constant.
- So-called "AK" model, and in its simplest form output does not even depend on labor.
- Results we would like to highlight apply with more general constant returns to scale production functions, e.g.

$$F\left[K\left(t\right), L\left(t\right), A\left(t\right)\right] = AK\left(t\right) + BL\left(t\right), \tag{40}$$

A First Look at Sustained Growth II

- Assume population grows at rate n as before (constant fertility rate [cfr] equation (32)).
- Combining with the production function (39),

$$\frac{\dot{k}(t)}{k(t)} = sA - \delta - n.$$

- Therefore, if $sA - \delta - n > 0$, there will be sustained growth in the capital-labor ratio.
- From (39), this implies that there will be sustained growth in output per capita as well.

A First Look at Sustained Growth III

Proposition (10) Consider the Solow growth model with the production function (39) $F\left[K\left(t\right), L\left(t\right), A\left(t\right)\right] = AK\left(t\right)$ and suppose that $sA - \delta - n > 0$. Then in equilibrium, there is sustained growth of output per capita at the rate $sA - \delta - n$. In particular, starting with a capital-labor ratio $k\left(0\right) > 0$, the economy has

$$k\left(t\right) = \exp\left(\left(sA - \delta - n\right)t\right)k\left(0\right)$$

and

$$y\left(t\right) = \exp\left(\left(sA - \delta - n\right)t\right)Ak\left(0\right).$$

- Note: no transitional dynamics.

Sustained Growth in Figure

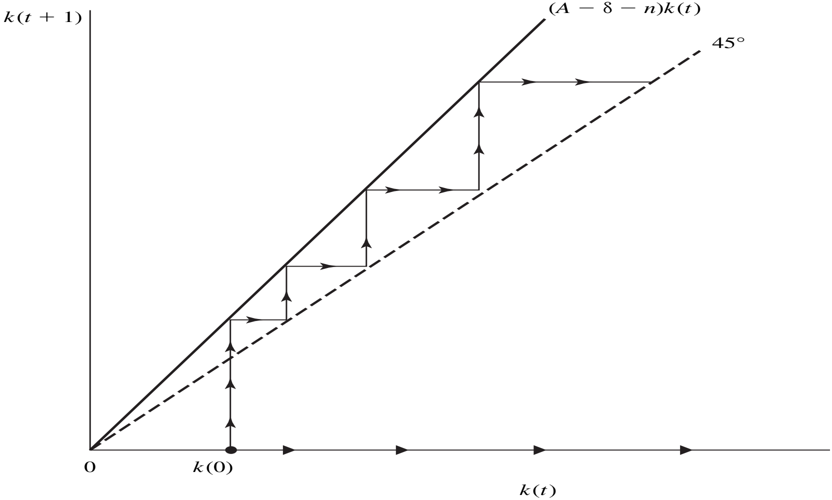


Figure 2.10 – Sustained growth with the linear AK technology with $sA - \delta - n > 0$.

A First Look at Sustained Growth IV

- Unattractive features:
  1. Knife-edge case, requires the production function to be ultimately linear in the capital stock.
  2. Implies that as time goes by the share of national income accruing to capital will increase towards 1. Empirically wages > 0, thus this would be impossible $R(t)K(t)/Y(t) \rightarrow 1$.
  3. Technological progress seems to be a major (perhaps the most major) factor in understanding the process of economic growth.

Section 7

Solow Model with Technological Progress

Balanced Growth I

- Production function $F\left[K\left(t\right), L\left(t\right), A\left(t\right)\right]$ is too general.
- May not have *balanced growth*, i.e. a path of the economy consistent with the *Kaldor facts* (Kaldor, 1963). K/Y, r, wL/Y and RK/Y are all (nearly) constants over time.
- Kaldor facts:
    - while output per capita increases, the capital-output ratio, the interest rate, and the distribution of income between capital and labor remain roughly constant.

1. $\dot{y}/y > 0$ does not decrease, y grows over time
2. k grows over time
3. R_t is nearly constant over time
4. k/y is nearly constant over time
5. wL/Y and RK/Y are nearly constant over time (Cobb-Douglas)
6. $\dot{y}/y$ differs across countries (but not time).

Historical Factor Shares

Figure 2.11 – Capital and Labor Share in the U.S. GDP.

Balanced Growth II

- Note capital share in national income is about $1/3$, while the labor share is about $2/3$.
- Ignoring land, not a major factor of production.
- But in poor countries land is a major factor of production.
- This pattern often makes economists choose $AK^{1/3}L^{2/3}$.
- Main advantage from our point of view is that balanced growth is the same as a steady-state in transformed variables
  - i.e., we will again have $\dot{k} = 0$, but the definition of k will change.
- But important to bear in mind that growth has many non-balanced features.
  - e.g., the share of different sectors changes systematically.
- Technological progress is Neutral if it does not alter the production function (P/N F/N) in a substantial way.

Types of Neutral Technological Progress I

- For some constant returns to scale function $\tilde{F}$:
  - *Hicks-neutral* technological progress:

$$\tilde{F}\left[K(t), L(t), A(t)\right] = A(t) F\left[K(t), L(t)\right],$$

    - Relabeling of the Isoquants (without any change in their shape) of the function $\tilde{F}\left[K(t), L(t), A(t)\right]$ in the L-K space.
  - *Solow-neutral* technological progress,

$$\tilde{F}\left[K(t), L(t), A(t)\right] = F\left[A(t) K(t), L(t)\right].$$

    - Capital-augmenting progress: Isoquants shifting with technological progress in a way that they have constant slope at a given labor-output ratio.
  - *Harrod-neutral* technological progress,

$$\tilde{F}\left[K(t), L(t), A(t)\right] = F\left[K(t), A(t) L(t)\right].$$

    - Increases output as if the economy had more labor: slope of the Isoquants are constant along rays with constant capital-output ratio.

Isoquants with Neutral Technological Progress

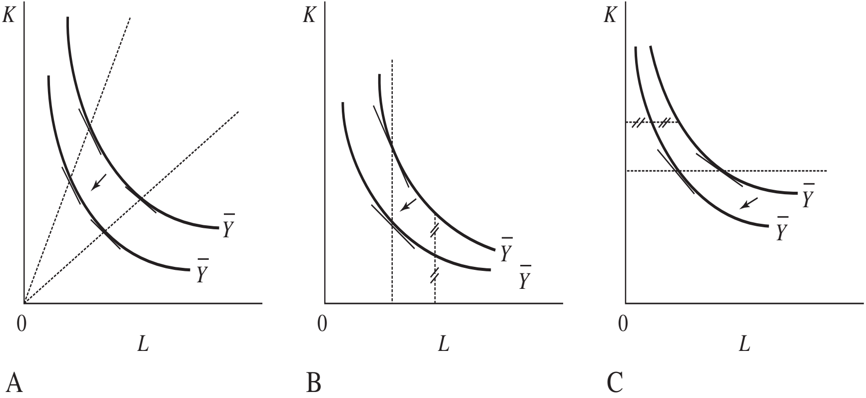


FIGURE 2.12 (A) Hicks-neutral, (B) Solow-neutral, and (C) Harrod-neutral shifts in isoquants.

Figure 2.12 – (A) Hicks-neutral, (B) Solow-neutral, and (C) Harrod-neutral shifts in

Types of Neutral Technological Progress II

- Could also have a vector valued index of technology $\mathbf{A}(t) = (A_H(t), A_K(t), A_L(t))$ and a production function

$$\tilde{F}[K(t), L(t), \mathbf{A}(t)] = A_H(t) F[A_K(t) K(t), A_L(t) L(t)], \quad (41)$$

- Nests the constant elasticity of substitution production function introduced in the Example above.
- But even (41) is a restriction on the form of technological progress, $A(t)$ could modify the entire production function.
- Balanced growth requires that all technological progress be labor augmenting or Harrod-neutral.

Cobb-Douglas

- Under Cobb-Douglas production function these forms are equivalent:

$$
\begin{aligned}
Y(t) &= AK(t)^{\alpha} L(t)^{1-\alpha}, \\
&= A^{1/2}\left[A^{1/2}K(t)\right]^{\alpha}\left[A^{1/2}L(t)\right]^{1-\alpha} \\
&= \left[A^{1/\alpha}K(t)\right]^{\alpha} L(t)^{1-\alpha} \\
&= K(t)^{\alpha}\left[A^{1/(1-\alpha)}L(t)\right]^{1-\alpha} \quad \text{: Harrod-neutral used}
\end{aligned}
$$

- Focus on continuous time models.

Balanced Growth and Euler's Theorem

- Key elements of balanced growth: constancy of factor shares and of capital-output ratio, $K(t)/Y(t)$. By factor shares, we mean:

$$\alpha_L(t) \equiv \frac{w(t)L(t)}{Y(t)} \quad \text{and} \quad \alpha_K(t) \equiv \frac{R(t)K(t)}{Y(t)}.$$

- By Assumption 1 and Euler Theorem $\alpha_L(t) + \alpha_K(t) = 1$.
- All variables should grow at a constant rate.

Uzawa's Theorem II

Theorem

(6) (Uzawa I) *Suppose* $L(t) = \exp(nt) L(0)$,

$$Y(t) = \tilde{F}(K(t), L(t), \tilde{A}(t)),$$

$\dot{K}(t) = Y(t) - C(t) - \delta K(t)$, *and* $\tilde{F}$ *is CRS in* K *and* L. *Suppose for* $t \geq \tau$, *where* $\tau < \infty$, $\dot{Y}(t) / Y(t) = g_Y > 0$, $\dot{K}(t) / K(t) = g_K > 0$ *and* $\dot{C}(t) / C(t) = g_C > 0$. *Then,*

1. $g_Y = g_K = g_C$; *and*
2. *for any* $t \geq \tau$, $\tilde{F}$ *can be represented as*

   $$Y(t) = F(K(t), A(t) L(t)),$$

   where $A(t) \in \mathbb{R}_+$, $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ *is homogeneous of degree 1, and*

   $$\dot{A}(t) / A(t) = g = g_Y - n.$$

Proof of Uzawa's Theorem I

- By hypothesis, for $t \geq \tau$, $Y(t) = \exp(g_Y(t-\tau))\, Y(\tau)$, $K(t) = \exp(g_K(t-\tau))\, K(\tau)$ and $L(t) = \exp(n(t-\tau))\, L(\tau)$ for some $\tau < \infty$.
- Since for $t \geq \tau$, $\dot{K}(t) = g_K K(t) = Y(t) - C(t) - \delta K(t)$, we have

$$(g_K + \delta)\, K(t) = Y(t) - C(t).$$

- Then,

$$\begin{aligned}(g_K + \delta)\, K(\tau) &= \exp((g_Y - g_K)(t-\tau))\, Y(\tau) \\ &\quad - \exp((g_C - g_K)(t-\tau))\, C(\tau)\end{aligned}$$

for all $t \geq \tau$.

Proof of Uzawa's Theorem II

- Differentiating with respect to time

$$
\begin{aligned}
0 = & \left(g_Y - g_K\right) \exp\left(\left(g_Y - g_K\right)\left(t - \tau\right)\right) Y\left(\tau\right) \\
& - \left(g_C - g_K\right) \exp\left(\left(g_C - g_K\right)\left(t - \tau\right)\right) C\left(\tau\right)
\end{aligned}
$$

  for all $t \geq \tau$.

- This equation can hold for all $t \geq \tau$
  1. if $g_Y = g_C$ and $Y\left(\tau\right) = C\left(\tau\right)$, which is not possible, since $g_K + \delta > 0$ and $K(\tau) > 0$.
  2. or if $g_Y = g_K$ and $C\left(\tau\right) = 0$, which is not possible, since $g_C > 0$ and $C\left(\tau\right) > 0$.
  3. or if $g_Y = g_K = g_C$, which must thus be the case.
- Therefore, $g_Y = g_K = g_C$ as claimed in the first part of the theorem.

Proof of Uzawa's Theorem III

- Next, the aggregate production function for time $\tau' \geq \tau$ and any $t \geq \tau$ can be written as

$$\begin{aligned} &\exp\left(-g_Y\left(t-\tau'\right)\right) Y\left(t\right) \\ &\quad = \tilde{F}\left[\exp\left(-g_K\left(t-\tau'\right)\right) K\left(t\right), \exp\left(-n\left(t-\tau'\right)\right) L\left(t\right), \tilde{A}\left(\tau'\right)\right]. \end{aligned}$$

- Multiplying both sides by $\exp\left(g_Y\left(t-\tau'\right)\right)$ and using the constant returns to scale property of F, we obtain

$$Y\left(t\right) = \tilde{F}\left[e^{\left(t-\tau'\right)\left(g_Y-g_K\right)} K\left(t\right), e^{\left(t-\tau'\right)\left(g_Y-n\right)} L\left(t\right), \tilde{A}\left(\tau'\right)\right].$$

- From part 1, $g_Y = g_K$, therefore

$$Y\left(t\right) = \tilde{F}\left[K\left(t\right), \exp\left(\left(t-\tau'\right)\left(g_Y-n\right)\right) L\left(t\right), \tilde{A}\left(\tau'\right)\right].$$

Proof of Uzawa's Theorem IV

- Moreover, this equation is true for $t \geq \tau$ regardless of τ', thus

$$\begin{aligned} Y(t) &= F\left[K(t), \exp\left(\left(g_Y - n\right)t\right)L(t)\right], \\ &= F\left[K(t), A(t)L(t)\right], \end{aligned}$$

with

$$\frac{\dot{A}(t)}{A(t)} = g_Y - n$$

establishing the second part of the theorem.

Implications of Uzawa's Theorem

In words: If the economy is a BGP (Balanced Growth Path) after $t \geq \tau$,

Corollary (3) Under the assumptions of Uzawa Theorem, after time τ technological progress can be represented as Harrod-Neutral (purely labor augmenting).

- Remarkable feature: stated and proved without any reference to equilibrium behavior or market clearing.
- Also, contrary to Uzawa's original theorem, not stated for a balanced growth path but only for an asymptotic path with constant rates of output, capital and consumption growth.
- **But**, not as general as it seems;
  - the theorem gives only one representation.

Stronger Theorem

Theorem

(7) (Uzawa's Theorem II) *Suppose that all of the hypothesis in Uzawa's Theorem are satisfied, so that* $\tilde{F} : \mathbb{R}_+^2 \times \mathcal{A} \to \mathbb{R}_+$ *has a representation of the form* $F(K(t), A(t)L(t))$ *with* $A(t) \in \mathbb{R}_+$ *and* $\dot{A}(t)/A(t) = g = g_Y - n$*. In addition, suppose that factor markets are competitive and that for all* $t \geq \tau$*, the rental rate satisfies* $R(t) = R^*$ *(or equivalently,* $\alpha_K(t) = \alpha_K^*$*). Then, denoting the partial derivatives of* $\tilde{F}$ *and* F *with respect to their first two arguments by* $\tilde{F}_K$*,* $\tilde{F}_L$*,* F_K *and* F_L*, we have*

$$\begin{aligned} \tilde{F}_K\left(K(t), L(t), \tilde{A}(t)\right) &= F_K(K(t), A(t)L(t)) \text{ and} \\ \tilde{F}_L\left(K(t), L(t), \tilde{A}(t)\right) &= A(t)F_L(K(t), A(t)L(t)). \end{aligned} \tag{42}$$

Moreover, if (42) holds and factor markets are competitive, then $R(t) = R^*$ *(and* $\alpha_K(t) = \alpha_K^*$*) for all* $t \geq \tau$*.*

Intuition

- Suppose the labor-augmenting representation of the aggregate production function applies.
- Then note that with competitive factor markets, as $t \geq \tau$,

$$\begin{aligned} \alpha_K(t) &\equiv \frac{R(t)K(t)}{Y(t)} \\ &= \frac{K(t)}{Y(t)} \frac{\partial F[K(t), A(t)L(t)]}{\partial K(t)} \\ &= \alpha_K^*, \end{aligned}$$

- Second line uses the definition of the rental rate of capital in a competitive market
- Third line uses that $g_Y = g_K$ and $g_K = g + n$ from Uzawa Theorem and that F exhibits constant returns to scale so its derivative is homogeneous of degree 0.

Intuition for the Uzawa's Theorems

- We assumed the economy features capital accumulation in the sense that $g_K > 0$.
- From the aggregate resource constraint, this is only possible if output and capital grow at the same rate.
- Either this growth rate is equal to n and there is no technological change (i.e., proposition applies with $g = 0$), or the economy exhibits growth of per capita income and capital-labor ratio.
- The latter case creates an asymmetry between capital and labor: capital is accumulating faster than labor.
- Constancy of growth requires technological change to make up for this asymmetry
- But this intuition does not provide a reason for why technology should take labor-augmenting (Harrod-neutral) form.
- But if technology did not take this form, an asymptotic path with constant growth rates would not be possible.

Interpretation

- Distressing result:
  - Balanced growth is only possible under a very stringent assumption.
  - Provides no reason why technological change should take this form.
- But when technology is endogenous, intuition above also works to make technology endogenously more labor-augmenting than capital augmenting.
- Only requires labor augmenting asymptotically, i.e., along the balanced growth path.
- This is the pattern that certain classes of endogenous-technology models will generate.

Implications for Modeling of Growth

- Does not require $Y(t) = F[K(t), A(t)L(t)]$, but only that it has a representation of the form $Y(t) = F[K(t), A(t)L(t)]$.
- Allows one important exception. If (Cobb-Douglas P/N F/N),

$$Y(t) = [A_K(t)K(t)]^{\alpha}[A_L(t)L(t)]^{1-\alpha},$$

  then both $A_K(t)$ and $A_L(t)$ could grow asymptotically, while maintaining balanced growth.
- Because we can define $A(t) = [A_K(t)]^{\alpha/(1-\alpha)} A_L(t)$ and the production function can be represented as

$$Y(t) = [K(t)]^{\alpha}[A(t)L(t)]^{1-\alpha}.$$

- Differences between labor-augmenting and capital-augmenting (and other forms) of technological progress matter when the elasticity of substitution between capital and labor is not equal to 1.

Further Intuition

- Suppose the production function takes the special form $F\left[A_K\left(t\right)K\left(t\right), A_L\left(t\right)L\left(t\right)\right]$.
- The stronger theorem implies that factor shares will be constant.
- Given constant returns to scale, this can only be the case when $A_K\left(t\right)K\left(t\right)$ and $A_L\left(t\right)L\left(t\right)$ grow at the same rate.
- The fact that the capital-output ratio is constant in steady state (or the fact that capital accumulates) implies that $K\left(t\right)$ must grow at the same rate as $A_L\left(t\right)L\left(t\right)$.
- Thus balanced growth can only be possible if $A_K\left(t\right)$ is asymptotically constant.

The Solow Growth Model with Technological Progress: Continuous Time I

- From Uzawa Theorem, production function must admit representation of the form

$$Y(t) = F[K(t), A(t)L(t)],$$

- Moreover, suppose

$$\frac{\dot{A}(t)}{A(t)} = g, \tag{43}$$

$$\frac{\dot{L}(t)}{L(t)} = n.$$

- Again using the constant saving rate

$$\dot{K}(t) = sF[K(t), A(t)L(t)] - \delta K(t). \tag{44}$$

The Solow Growth Model with Technological Progress: Continuous Time II

- Now define $k(t)$ as the *effective capital-labor* ratio, i.e.,

$$k(t) \equiv \frac{K(t)}{A(t)\,L(t)}. \tag{45}$$

- Slight but useful abuse of notation.
- Differentiating this expression with respect to time,

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - g - n. \tag{46}$$

- Output per unit of effective labor can be written as

$$\begin{aligned} \hat{y}(t) &\equiv \frac{Y(t)}{A(t)\,L(t)} = F\left[\frac{K(t)}{A(t)\,L(t)}, 1\right] \\ &\equiv f(k(t)). \end{aligned}$$

The Solow Growth Model with Technological Progress: Continuous Time III

- Income per capita is $y(t) \equiv Y(t) / L(t)$, i.e.,

$$\begin{aligned} y(t) &= A(t)\,\hat{y}(t) \\ &= A(t)\,f(k(t)). \end{aligned} \tag{47}$$

- Clearly if $\hat{y}(t)$ is constant, income per capita, $y(t)$, will grow over time, since $A(t)$ is growing.
- Thus should not look for "steady states" where income per capita is constant, but for *balanced growth paths*, where income per capita grows at a constant rate.
- Some transformed variables such as $\hat{y}(t)$ or $k(t)$ in (46) remain constant.
- Thus balanced growth paths can be thought of as steady states of a transformed model.

The Solow Growth Model with Technological Progress: Continuous Time IV

- Hence use the terms "steady state" and balanced growth path interchangeably.
- Substituting for $\dot{K}(t)$ from (44) into (46):

$$\frac{\dot{k}(t)}{k(t)} = \frac{sF[K(t), A(t)L(t)]}{K(t)} - (\delta + g + n).$$

- Now using (45),

$$\frac{\dot{k}(t)}{k(t)} = \frac{sf(k(t))}{k(t)} - (\delta + g + n), \tag{48}$$

- Only difference is the presence of g: k is no longer the capital-labor ratio but the *effective* capital-labor ratio.

The Solow Growth Model with Technological Progress: Continuous Time V

Proposition (11) Consider the basic Solow growth model in continuous time, with Harrod-neutral technological progress at the rate g and population growth at the rate n. Suppose that Assumptions 1 and 2 hold, and define the effective capital-labor ratio as in (45). Then there exists a unique steady state (balanced growth path) equilibrium where the effective capital-labor ratio is equal to $k^* \in (0, \infty)$ and is given by

$$\frac{f(k^*)}{k^*} = \frac{\delta + g + n}{s}. \tag{49}$$

Per capita output and consumption grow at the rate g.

The Solow Growth Model with Technological Progress: Continuous Time VI

- Equation (49), emphasizes that now total savings, $sf(k)$, are used for replenishing the capital stock for three distinct reasons:
  1. depreciation at the rate δ.
  2. population growth at the rate n, which reduces capital per worker.
  3. Harrod-neutral technological progress at the rate g.
- Now replenishment of effective capital-labor ratio requires investments to be equal to $(\delta + g + n)k$.

Solow Growth w/ Technological Progress: Continuous Time VII

Proposition (12) Suppose Assumptions 1 and 2 hold and let $A(0)$ be the initial level of technology. Denote the balanced growth path level of effective capital-labor ratio by $k^*(A(0), s, \delta, n)$ and the level of output per capita by $y^*(A(0), s, \delta, n, t)$. Then

$$\begin{aligned}
\frac{\partial k^*(A(0), s, \delta, n)}{\partial A(0)} &= 0, \ \frac{\partial k^*(A(0), s, \delta, n)}{\partial s} > 0, \\
\frac{\partial k^*(A(0), s, \delta, n)}{\partial n} &< 0 \text{ and } \frac{\partial k^*(A(0), s, \delta, n)}{\partial \delta} < 0,
\end{aligned}$$

and also for each t

$$\begin{aligned}
\frac{\partial y^*(A(0), s, \delta, n, t)}{\partial A(0)} &> 0, \ \frac{\partial y^*(A(0), s, \delta, n, t)}{\partial s} > 0, \\
\frac{\partial y^*(A(0), s, \delta, n, t)}{\partial n} &< 0 \text{ and } \frac{\partial y^*(A(0), s, \delta, n, t)}{\partial \delta} < 0.
\end{aligned}$$

The Solow Growth Model with Technological Progress: Continuous Time VIII

Proposition (13) Suppose that Assumptions 1 and 2 hold, then the Solow growth model with Harrod-neutral technological progress and population growth in continuous time is asymptotically stable, i.e., starting from any $k(0) > 0$, the effective capital-labor ratio converges to a steady-state value k^* ($k(t) \to k^*$).

- Now model generates growth in output per capita, but entirely *exogenously*.

Section 8

Comparative Dynamics

Comparative Dynamics I

- Comparative dynamics: dynamic response of an economy to a change in its parameters or to shocks.
- Different from comparative statics in Propositions above in that we are interested in the entire path of adjustment of the economy following the shock or changing parameter.
- For brevity we will focus on the continuous time economy.
- Recall
$$\dot{k}(t)/k(t) = sf(k(t))/k(t) - (\delta + g + n)$$

Comparative Dynamics in Figure

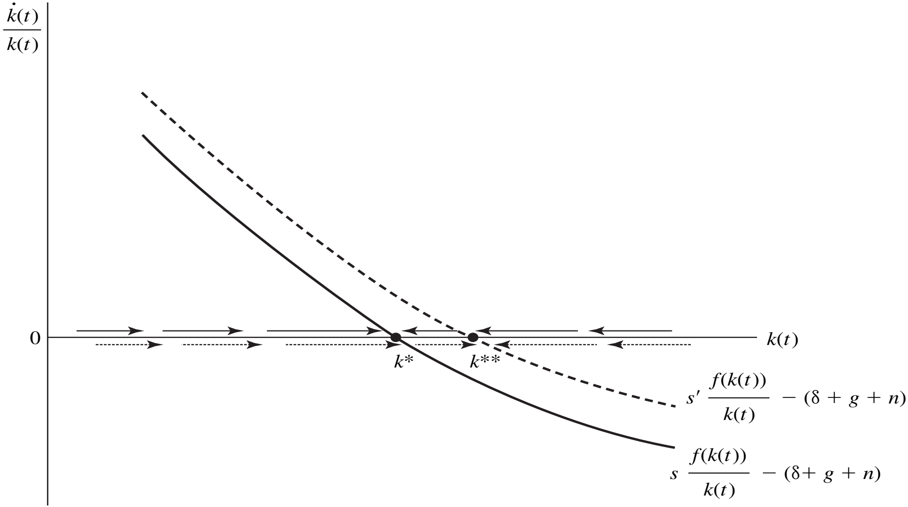


Figure 2.13 – Dynamics following an increase in the savings rate from s to s'. The solid arrows show the dynamics for the initial steady state, while the dashed arrows show the dynamics for the new steady state.

Comparative Dynamics II

- One-time, unanticipated, permanent increase in the saving rate from s to s'.
    - Shifts curve to the right as shown by the dotted line, with a new intersection with the horizontal axis, k^{**}.
    - Arrows on the horizontal axis show how the effective capital-labor ratio adjusts gradually to k^{**}.
    - Immediately, the capital stock remains unchanged (since it is a *state* variable).
    - After this point, it follows the dashed arrows on the horizontal axis.
- s changes in unanticipated manner at $t = t'$, but will be reversed back to its original value at some known future date $t = t'' > t'$.
    - Starting at t', the economy follows the rightwards arrows until t'.
    - After t'', the original steady state of the differential equation applies and leftwards arrows become effective.
    - From t'' onwards, economy gradually returns back to its original balanced growth equilibrium, k^*.

Section 9

Conclusions

Conclusions

- Simple and tractable framework, which allows us to discuss capital accumulation and the implications of technological progress.
- Solow model shows us that if there is no technological progress, and as long as we are not in the AK world, there will be no sustained growth.
- Generate per capita output growth, but only exogenously: technological progress is a black-box.
- Capital accumulation: determined by the saving rate, the depreciation rate and the rate of population growth. All are exogenous.
- Need to dig deeper and understand what lies in these black boxes.